DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA 93943

DECOMPOSITION OF DUAL-ANGULAR INTEGER PROGRAMS, WITH APPLICATIONS TO MULTI-STAGE CAPITAL BUDGETING

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF OPERATIONS RESEARCH

AND THE COMMITTEE ON GRADUATE STUDIES

OF STANFORD UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

Kevin John Reardon

July 1974

T170828

Thesis
R2545

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA 93940

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Approved for the University Committee
on Graduate Studies:

ACKNOWLEDGMENTS

I extend my sincere appreciation to Professor Frederick S. Hillier who introduced me to the fields of integer programming and capital budgeting, and who served as my principal research advisor. Professor Hillier's active interest, constant encouragement and expert guidance contributed substantially to the successful completion of this dissertation. I am also indebted to Dr. John A. Tomlin for his many helpful suggestions on the integer programming portion of my work, and to Professor Gerald J. Lieberman for his interest and guidance during my four years at Stanford.

The opportunity and support for my graduate education was provided by the Office of the Chief of Naval Operations under the Junior Line Officer Advanced Educational Program (Burke Program), and is gratefully acknowledged. Additional support during the final year of my studies was funded by the Office of Naval Research and the National Science Foundation.

I have become increasingly aware of the importance of my parents' advice and encouragement over the years, and gratefully share credit with them for this dissertation. To my wife, Sue, who put up with odd study hours and unmowed lawns, and whose loving optimism helped keep me going, I owe much more than words can express.

A special note of thanks goes to Miss Gail Lemmond for her excellent job of drafting figures and typing from an occasionally illegible manuscript.

TABLE OF CONTENTS

CHAPTER		PAGE
	ACKNOWLEDGMENTS.	iii
1	INTRODUCTION AND DISCUSSION OF CAPITAL BUDGETING	1
	1.1 Historical Perspective.	2
	1.2 The Capital Rationing Problem	6
	1.3 Capital Budgeting with Borrowing.	10
	1.4 Chance-Constrained Programming Formulation.	12
2	THE MULTI-STAGE MODEL.	17
	2.1 Basis for the Model	17
	2.2 The Basic Two-Stage Model	18
	2.3 Extension for Uncertain Future Opportunities. . . .	23
3	STRUCTURE AND SOLUTION OF THE MODEL.	29
	3.1 Size and Structure Aspects.	29
	3.2 Inclusion of Borrowing/Lending and Fractional Investments	35
	3.3 Summary	40
4	THE DECOMPOSITION METHOD: GENERAL DISCUSSION.	42
	4.1 Applications of the Method.	42
	4.2 Motivation for the Method	44
	4.3 Description of the Method	46
5	THE DECOMPOSITION METHOD: IMPLEMENTATION.	51
	5.1 Introduction.	51
	5.2 Penalties	52
	5.3 Surrogate Constraints	59
	5.4 Steps of the Algorithm.	62
	5.5 Possible Modifications to the Algorithm	72
6	COMPUTATIONAL RESULTS.	75
	6.1 Computer Code	75
	6.2 Description of Testing Procedure.	77
	6.3 Test Problem Number One	79
	6.4 Test Problem Number Two	83
	6.5 Test Problems Three and Four.	84
	6.6 Test Problems Five and Six.	91
	6.7 Test Problem Number Seven	91
	6.8 Schrage's Test Problem.	96
	6.9 Summary of Computational Results.	100

CHAPTER PAGE

7 CONCLUSIONS AND AREAS FOR FUTURE RESEARCH. 103
7.1 Capital Budgeting Formulation 103
7.2 Subproblem-Decomposition Method 105
7.3 Areas for Future Research 106

APPENDIX 109

REFERENCES 122

CHAPTER 1

INTRODUCTION AND DISCUSSION OF CAPITAL BUDGETING

One of the most important recurring decisions for any economic unit, public or private, is that of allocating its limited financial resources in a manner which best supports the attainment of its goals. Nearly always, such decisions must be made in an environment characterized by incomplete information, uncertainty, complex interactions among activities, imperfect capital markets, and many other complicating factors. The problem is compounded by the fact that most organizations in government and private business continue to perform their functions over a number of years, and must therefore consider the possible future impacts of their decisions. In fact, it is often necessary not only to resolve those issues on which immediate action is required, but to integrate such current decisions into a strategic plan which applies far into the future.

During the past quarter-century, managerial economists have given considerable attention to this "capital budgeting" problem, and have greatly improved the methods used to deal with it. The theory of capital budgeting is by no means complete, however, and there remain many unanswered questions of practical significance. The development of techniques for constructing and solving mathematical models of complex decision processes has allowed explicit consideration of interactions among investment alternatives and other complicating factors. Such mathematical programming techniques have provided a

more theoretically-sound basis for decision making than was previously available, and the computational efficiency with which they may often be solved shows great promise for the resolution of large-scale capital budgeting problems.

This dissertation describes and comments upon several formulations of the capital budgeting problem which have appeared in the literature of mathematical programming, and presents an extended multi-stage model for the selection and timing of investment proposals whose future availability is uncertain. The model is developed in the context of pure and mixed integer linear programming, with particular attention given to the special structure of its constraint set. We then present an integer linear programming method designed to exploit special structures of this type, commonly known as "dual-angular", by decomposition into groups of smaller subproblems. (Such dual-angular constraint matrices also arise in a number of other applications not directly related to multi-stage capital budgeting.) The decomposition method, implemented within the framework of a penalty-based zero-one integer programming algorithm, has enjoyed considerable success in reducing the effort required for solving a variety of test problems. A detailed discussion of this specific implementation and of its associated computational experience is also presented.

1.1. Historical Perspective

Prior to the publication of Lorie and Savage [1955], most theoretical techniques for selection of investment proposals were

concerned with the analysis of individual projects. Acceptance or rejection of any given opportunity was based mainly on the value of some measure of merit, which was calculated with limited consideration of physical and budgetary interactions with other opportunities and the current operations of the firm.[1] Such a procedure can result in an optimum choice of projects only if all those under consideration are completely independent of one another.[2] This precludes such commonplace interrelationships as competition for the same scarce resources (wealth, manpower, customers, etc.) or the sharing of costs for new facilities required by two or more projects.

A classic example of such techniques for evaluating individual projects is the payback criterion. Using this approach, a proposal is deemed acceptable if its total earnings meet or exceed the required initial capital outlay within a specified period of time. The determination of the "payback period" is rather arbitrary, with management often requiring that those projects which it considers more "risky" than others be repaid in a shorter period of time. In addition, different types of investments are frequently assigned different basic payback guidelines which are then "risk-adjusted". Some of the disadvantages associated with this approach are: (1) cash flows (positive or negative) which occur after the specified payback period are completely disregarded in the evaluation; (2) the salvage value of a project which may be terminated beyond its payback limit is not considered; (3) the value of a dollar received at any time

[1] See Dean [1951] and Klevorick [1969] for additional discussions.

[2] This has been discussed by Lorie and Savage [1955], Hirshleifer [1958], Weingartner [1967], and Klevorick [1969].

during the payback period is considered to be a constant, even though the value of money to the firm may vary within that period; (4) if different payback requirements are levied on different projects, there arises the situation wherein a dollar received from project A after its payback period has expired is given no value, but the same dollar received from project B at the same time is assigned a positive utility; and (5) the criterion provides little guidance when budgetary constraints are imposed on several periods.

While not advocating payback as a criterion for selecting capital investments, Weingartner [1971] has discussed in detail its apparent attractiveness to businessmen. Payback was presented variously as a measure of rate of return, as a form of break-even analysis, as a measure of the "liquidity" of a capital asset, and as a measure of the rate of resolution of uncertainty. Finally, it was concluded that payback was an oversimplification in each of the above cases, but that further study of its popularity might be a prerequisite to the development of superior alternatives.

Another well-known technique for evaluating individual projects is the internal rate of return rule. This rule requires the calculation for each project of that rate, r, which would cause the project's cash flows over future years to sum to zero if they were first discounted by r. That is, a solution, r, is sought for the polynomial equation

$$(P)\quad a_1 + \frac{a_2}{1+r} + \frac{a_3}{(1+r)^2} + \cdots + \frac{a_T}{(1+r)^{T-1}} = 0 \quad ,$$

where a_i is the net cash flow associated with the project in period i. The procedure then requires that the proposals be ranked in order of decreasing internal rate of return, and that all those with rates exceeding a predetermined "rejection rate" (e.g., "cost of capital") be accepted. This method suffers from several shortcomings: (1) the internal rate of return as defined above may not be unique, or the only roots of the polynomial (P) may be complex; (2) the cash flows in every period are given the same relative value in calculating r, even though the relative value of money to the firm may vary from period to period; (3) the "cost of capital" is an extremely elusive figure whose determination has received much attention in the financial literature[3]; and (4) the selection criterion provides insufficient guidance when the choice of projects is restricted by constraints on total cash flow in several periods.

The net present value rule is similar to the internal rate of return rule, except that the former's discount rate is externally-supplied (often by a measure of the "cost of capital") instead of being determined from the values of the cash flow components. In fact, the applicable discount rate may vary from one period to another when using the former rule, whereas the latter uses a uniform rate. This allows for explicit use of information regarding future financial trends; however, the firm is still faced with the difficult task of determining the proper discount factor for each period. Under the assumptions of a perfect capital market and complete independence among projects, use of the borrowing rate for discounting and selection

[3] See Hirshleifer [1958] or Klevorick [1969], for example.

of all projects with positive present values will lead to a solution which maximizes the net present value of the firm. Unfortunately, relaxation of either of these restrictive assumptions requires modification of the selection rules and may well render the rules non-optimal. In particular, if multi-period budgetary constraints are imposed, it becomes theoretically impossible to determine the appropriate discount rates and the aforementioned selection rules would fail even if the rates were available. (For a detailed discussion of the problems involved in searching for the proper discount rates, the interested reader is referred to Hirschleifer [1958].)

1.2. The Capital Rationing Problem

Lorie and Savage [1955] presented a capital budgeting model designed to overcome some of the limitations of the then generally-accepted internal rate of return criterion for project selection.[4] The problem was to select a portfolio of investment projects which would have maximum total present value, subject to upper bounds on total expenditures in several periods and an exogenously-determined value for the firm's cost of capital in each period. Thus, rather than evaluating individual projects on their own merits and making independent investment decisions, the objective was to make a concurrent selection which explicitly considered the budgetary interactions of the available projects. Assuming that all cash flows were known with certainty and that the projects were mutually independent, Lorie and Savage were able to make some progress toward their goal when budgetary

[4] See Dean [1951].

constraints applied in only one or two periods and fractional investments were allowed. Their iterative method was a form of "generalized Lagrange multiplier" technique, as discussed by Everett [1963].

In a highly significant 1962 doctoral dissertation, Weingartner [1967] showed that the Lorie-Savage problem could be formulated as a linear program as follows:

$$
\begin{aligned}
(\text{LP}) \qquad \text{maximize} \quad & \sum_j b_j x_j \,, \\
\text{subject to} \quad & \sum_j a_{jt} x_j \leq M_t, \qquad \text{for} \quad t = 1, \ldots, T, \\
& 0 \leq x_j \leq 1 \qquad \text{for all} \quad j \,,
\end{aligned}
$$

where

x_j is the level of participation in project j,

b_j is the net present value of project j,

a_{jt} is the gross expenditure required for project j in time period t, and

M_t is the spending limit which applies in period t.

Recognition of this formulation allowed application of the powerful theoretical and computational techniques of linear programming. In particular, Weingartner presented an extensive analysis of the economic interpretations of duality theory as applied to (LP) and various extensions.

The extensions to (LP) which were discussed included a change in the objective to maximize the worth of the firm at some future time (the "horizon"), and allowances for borrowing and lending under several

types of capital market imperfections. Although some types of investment opportunities are highly divisible and may therefore be undertaken at levels corresponding to fractional values of their associated decision variables (e.g., borrowing or lending), most capital budgeting decisions are of the accept-reject variety for which indications of fractional participation have little or no meaning. By appending to (LP) the additional constraint

$$(*) \qquad x_j = 0 \text{ or } 1, \qquad \text{for } j \in S,$$

one may exclude the possibility of obtaining such indications for those projects belonging to the set S.

Also, as Weingartner pointed out, this integer linear programming formulation can be used to express various types of project interdependencies by use of additional constraints. For example, the constraint

$$x_k + x_\ell \leq 1 \qquad \text{where } k, \ell \in S$$

together with (*) indicates that projects k and ℓ are mutually exclusive, and

$$x_m - x_n \leq 0, \qquad \text{where } m, n \in S$$

together with (*) indicates that acceptance of project m requires acceptance of project n. Unfortunately, the solution of an integer linear program is a much more difficult task than that of the corresponding

linear program in which the integer restrictions are removed; furthermore, the most successful (enumerative) techniques for integer linear programming are not readily amenable to duality analysis or sensitivity analysis.

Baumol and Quandt [1965] expressed several criticisms of Weingartner's formulation of the capital rationing problem. They considered the most serious shortcoming to be the previously-mentioned difficulty of determining an appropriate discount rate for each period, and indicated that other writers had unjustifiably glossed-over the problem. In addition, they were dissatisfied with the fact that Weingartner's model did not allow net cash inflows from selected projects to be used in the same periods for augmenting the budgeted amounts. Baumol and Quandt argued that a firm should be able to reinvest such funds, and that it should be concerned with maximizing the utility of the funds withdrawn from its portfolio, rather than with maximizing the present value of that portfolio. They further claimed that such an approach would have the additional advantage of obviating the need for determining discount rates for each period, and proposed the following model:

$$\text{maximize} \qquad \sum_t U_t W_t$$

$$\text{subject to} \qquad -\sum_t a_{jt} + W_t \leq M_t , \qquad \text{for } t = 1, \ldots, T,$$

$$x_j \geq 0 , \qquad \text{for all } j,$$

$$W_t \geq 0 , \qquad \text{for all } t,$$

where U_t is the utility of one dollar of withdrawal in period t, W_t is the amount withdrawn in period t, and a_{jt} is the net cash flow from project j in period t.[5]

Weingartner [1966b] responded to the Baumol-Quandt paper with an admission of the difficulty involved with discount rates and a defense of his decision not to allow reinvestment of earnings. He also pointed out several faults in the Baumol-Quandt model, and suggested a third model which sought to maximize dividend growth. Following this exchange, several other papers appeared with suggestions on formulating an appropriate objective function for the capital rationing problem.[6] It appears, however, that the controversy remains unresolved, and quite possibly may never by completely settled in a way that is both theoretically sound and computationally tractable. Meanwhile, advocates of programming approaches to capital budgeting have tended to choose particular objective measures (recognizing their imperfections) and to proceed to develop models and techniques which can improve current practices. The most widely-accepted objective criterion is that of maximizing total net present value from the projects selected, and that is the measure which will be used in the remainder of this study.

1.3. Capital Budgeting with Borrowing

Before proceeding to a dicussion of the incorporation of risk in mathematical programming models and to the presentation of our proposed multi-stage model, we should first show the extension of

[5] The authors stated that the use of a non-linear utility function would add nothing to their argument.

[6] Carleton [1969], Lusztig and Schwab [1968], Mao [1969], and Bernhard [1969].

Weingartner's deterministic formulation to allow for borrowing of funds. This capital rationing model can be readily altered to allow the firm to borrow funds under various types of capital market conditions. The budgetary constraints remain in the model, but are now interpreted as representing internally-generated funds which may be supplemented at a cost from external sources. We can also assume that surplus funds in any period may be lent out at the prevailing lending rate of interest, which may differ from the borrowing rate. Assuming unlimited ability to borrow and lend on one-period contracts at constant rates i_B and i_L, respectively, and using i_B for the discount factor, the model becomes:

$$\text{maximize} \quad \sum_j b_j x_j - i_B \sum_t \frac{u_t}{(1+i_B)^t} + i_L \sum_t \frac{v_t}{(1+i_B)^t} ,$$

$$\text{subject to} \quad \sum_j a_{j1} x_j - u_1 + v_1 \leq M_1 ,$$

$$\sum_j a_{jt} x_j + (1+i_B) u_{t-1} - (1+i_L) v_{t-1} - u_t + v_t \leq M_t ,$$

$$\text{for} \quad t = 2, \ldots, T ,$$

$$0 \leq x_i \leq 1 , \qquad \text{for all} \quad i,$$

$$u_t, v_t \geq 0, \qquad \text{for} \quad t = 1, \ldots, T ,$$

where u_t and v_t are the amounts borrowed and lent, respectively, in period t.

We can introduce various market imperfections by adding additional constraints to the above formulation. For example, the provision for

unlimited borrowing can be replaced by upper bounds, B_t, on the amounts borrowed in each period:

$$u_t \leq B_t, \qquad \text{for } t = 1, \ldots, T.$$

Such restrictions might be applied to a division of a firm by its home office, or may arise for the entire firm due to liquidity considerations or limitations imposed by lending agents.

Another realistic provision we might make is for the borrowing rate to be an increasing function of the amount borrowed. This is easily accomplished in any period by replacing u_t with variables u_{tk}, associating a different interest rate with each value of k, and placing an upper bound on the amount that may be borrowed under each rate. This extension results in a linear program with a separable convex objective, which can be readily solved with the simplex method. Weingartner [1967] has also extended his basic model to consider the amount and timing of equity financing.

1.4. Chance-Constrained Programming Formulation

Up to this point, we have been concerned only with the formulation and analysis of the deterministic capital budgeting problem. We have assumed that the decision-maker knows with complete certainty what projects will be available for selection, what profits and cash-flow patterns will be associated with these projects, what limitations will be applied to expenditures and borrowing, and what the prevailing rates

of interest will be during that portion of the future with which he is concerned. Although the solution of a problem under these assumptions can provide valuable insight into the implications of various constraints and the financial and physical interactions among projects, "real world" decisions require the incorporation of the firm's attitude toward risk -- the possibility that investments may not turn out as well as expected.

In order to incorporate this concept of risk into the analysis, one must first identify those factors which can significantly affect the solution when they are varied within their ranges of possible values. The next step is to estimate probability distributions on these most sensitive variables[7], and to set all others at nominal (perhaps most likely, or mean) values. One can then employ a measure of the desirability of a project portfolio based on the expected results and the attendant risk, and proceed to solve and analyze the problem.

Many different methods exist for decision-making under risk, but the two which have received the most attention in the capital budgeting literature are the expected-utility approach and chance-constrained programming. Maximization of expected utility has been a popular objective among authors in the related area of portfolio selection[8], and has been applied in recent years to capital budgeting by Hillier [1969] and Klevorick [1969]. Its main difficulties lie in the complicated utility functions (cubic, exponential, etc.) which arise, and in the amount of work necessary to adequately determine these functions. The chance-constrained programming approach seeks to avoid these difficulties by using a simpler objective function and incorporating

[7] See Hillier [1963] and [1965].

[8] Markowitz [1959].

risk-aversion into the constraints of the problem. This is done by requiring that any feasible selection of projects have a probability of at least α_i of satisfying the ith constraint.[9] Although several types of objective criteria have been proposed for chance-constrained programming, the most common one is to maximize the expected value of the chosen measure-of-merit, such as net present value.

Assuming a long-run perfect capital market, Naslund [1967, 1971] presented and discussed a chance-constrained capital budgeting model in a manner which closely paralleled Weingartner's deterministic development. He showed how chance-constraints led to the requirement for a specific financial reserve, or "cushion", in each period, with the amount of that reserve depending on the amount of risk considered acceptable. Naslund also developed acceptance criteria for projects and considered the effects of changes in interest rates and variance of returns. The model -- a direct extension of Weingartner's horizon posture model -- was as follows:

$$\text{maximize} \qquad E\Big(\sum_i \hat{d}_i\, x_i + v_T - w_T\Big) ,$$

$$\text{subject to} \qquad \text{Prob}\Big(\sum_i d_{i1}\, x_i + v_1 - w_1 \le D_1\Big) \ge \alpha_1 ,$$

$$\text{Prob}\Big(\sum_{j=1}^{T} \sum_i d_{ij}\, x_i - \sum_{j=1}^{t-1} v_j r + \sum_{j=1}^{t-1} w_j r + v_t - w_t \le \sum_{j=1}^{t} D_j\Big) \ge \alpha_t ,$$

$$\text{for } t = 2, \ldots, T,$$

$$0 \le x_i \le 1 , \qquad \text{for all } i,$$

$$u_t,\ v_t \ge 0 , \qquad \text{for } t = 1, \ldots, T,$$

[9] See Hillier and Lieberman [1967], pp. 536-42, for an introduction to chance-constrained programming.

where

E is the expected value operator,

D_t is the amount of money supplied externally in period t,

d_{it} is the random variable representing net cash flow from project i in period t,

v_t is the amount lent in period t,

w_t is the amount borrowed in period t,

r is the borrowing, lending, and discount rate,

x_i is the fraction of project i undertaken,

$\hat{d}_i$ is the expected horizon value of all post-horizon cash flows from project i,

T is the horizon time.

In this case, the probabilistic budgetary constraints are imposed on the cumulative net cash flows as adjusted for interest received or paid out. It is assumed that borrowing and lending take place on yearly contracts, and that these contracts must be negotiated prior to knowing the outcomes in the respective periods. Therefore, it is not possible to guarantee with certainty that the cash flow constraints will be satisfied under all conditions.

Assuming that the random variables, d_{ij}, are independently normally-distributed with mean μ_{ij} and variance σ^2_{ij}, we can derive a deterministic equivalent of the model wherein all investment and financing decisions are made at the outset (a zero-order decision rule which does not explicitly contain the random elements d_{ij}):

$$\text{maximize} \quad \sum_i \hat{d}_i x_i + v_T - w_T ,$$

$$\text{subject to} \quad \sum_i \mu_{i1} x_i + v_1 - w_1 + \sqrt{\left(\sum_i \sigma_{i1}^2 x_i^2\right)}\, F^{-1}(\alpha_1) \leq D_1$$

$$\sum_{j=1}^{t} \sum_i \mu_{ij} x_i - \sum_{j=1}^{t-1} v_j r + \sum_{j=1}^{t-1} w_j r + v_t - w_t$$

$$+ \sqrt{\left(\sum_i \sigma_{it}^2 x_i^2\right)}\, F^{-1}(\alpha_t) \leq \sum_{j=1}^{t} D_j , \qquad \text{for } t = 2, \ldots, T,$$

$$0 \leq x_i \leq 1 , \qquad \text{for all } i ,$$

$$u_t, v_t \geq 0 , \qquad \text{for } t = 1, \ldots, T$$

where $F(\cdot)$ represents the appropriate distribution function for each constraint.

Had the random variables, d_{ij}, not been independent, the deterministic equivalent constraints would have included covariance terms under their radicals. As Naslund pointed out, the model can easily be extended to allow the D_j to be normally-distributed random variables. Constraints expressing contingency and mutual exclusion relationships may, of course, be added as necessary.

The terms above which involve square roots represent the financial slacks, or "cushions", which must be added to the expected net cash flows as a precaution against the possibility that the actual realized flows might violate the budget constraints. As intuition would indicate, the sizes of these financial slacks are increasing functions of the cash flow variances and the required probabilities, α_i.

CHAPTER 2

THE MULTI-STAGE MODEL

2.1. Basis for the Model

The model which we will propose in this chapter is an extension of the "basic two-stage model" for chance-constrained capital budgeting which was presented in Hillier [1969]. Hillier's model was designed to facilitate a strategy approach to the selection of future projects as well as those currently available. In the two-stage context, certain decisions are made immediately (stage 1) with consideration given to investment opportunities which will become available in the future, and then a second set of decisions will be made in the future (stage 2) after having observed some of the results from the first set. This type of formulation not only can provide guidance on which future investments should be selected, given specific outcomes from the first group, but also can indicate whether the previously selected projects should be reduced, continued, or expanded. In general, the model can be set up to handle any number of stages, but for purposes of exposition we will confine most of our discussion to the two-stage case.

Using the notation of Hillier [1969], the general problem is to determine a decision vector, $\delta = (\delta_1, \ldots, \delta_m)$, whose zero-one elements indicate the rejection or acceptance, respectively, of their associated projects. The determination is to be made so as to maximize the expected net present value of the portfolio, subject to probabilistic constraints on net cash flow in each period and on cumulative net cash flow through

each period. In addition, it is required that δ be a member of the set of feasible decision vectors, designated by S. This set of vectors may be limited by mutual exclusion, or contingency relationships among projects, and may also reflect additional constraints of a budgetary or non-budgetary nature. We further assume that the arguments of the probability functions and the restrictions on S can be expressed as linear inequalities.

In notational form, the problem is to

$$\text{maximize} \qquad E\{P(\delta)\} = \sum_{j=1}^{m} E(P_j)\delta_j \ ,$$

$$\text{subject to} \qquad \text{Prob}\{(NCF)_k \geq L_k\} \geq \alpha_k, \qquad \text{for } k = 1, \ldots, n,$$

$$\text{Prob}\{(CNCF)_k \geq C_k\} \geq \eta_k, \qquad \text{for } k = 2, \ldots, n,$$

$$\delta \in S \ , \ \delta_j = 0 \text{ or } 1 \qquad \text{for } j = 1, \ldots, m,$$

where $P(\delta)$ is the net present value of the portfolio, P_j is the net present value of project j, $(NCF)_k$ is the net cash flow from the projects in period k, $(CNCF)_k$ is the cumulative net cash flow through period k, L_k and C_k are the bounds imposed in period k, and α_k and η_k are the specified risk levels in period k.

2.2. The Basic Two-Stage Model

Hillier's "basic two-stage model", which was mentioned previously, can be presented as a specific form of the general problem stated above.

We augment δ with another decision vector, $\Delta = (\Delta_1, \ldots, \Delta_{m'})$, defined in the same way, where each Δ_i indicates the acceptance or rejection of a project available at the beginning of the second stage. These second stage "projects" may include new investment opportunities as well as possible changes in the commitments which were made at the outset. Examples of the latter possibility might be to increase the funding of a research and development effort undertaken at stage one, or to undertake at stage two a project which was previously rejected but is still available. Since these alternatives require previous acceptance and rejection, respectively, we would have to include the appropriate contingency and mutual exclusion constraints in the definition of the feasible region for (δ,Δ), which we designate by S'.

The additional information which becomes available at the end of the first stage will normally include the values of the first-period cash flows from the accepted stage-one projects, and may also include the values of economic or business indicators such as the gross national product or Dow Jones Industrial Average. By dividing the set of all possible outcomes of the relevant additional information into a manageable number of mutually-exclusive and exhaustive subsets, and associating probabilities of occurrence with these subsets, we can develop a single optimum specification of δ and a different specification of Δ for each subset. This is the strategy approach to which we alluded earlier -- the solution procedure specifies a group of projects to be undertaken immediately, and also delineates a plan for subsequent action based on the initial results from these projects.

Using Hillier's notation, we can describe the subdivision of the outcome set in more formal terms as follows: Let V be the random vector whose elements are the random variables of interest which will be observed at the end of the first period, and let n_v be the number of elements in V. Let R be the range space of V, so that R is that subset of n_v-dimensional Euclidean space whose elements are all the possible values of V. This set, R, is partitioned into n_r mutually-exclusive and exhaustive subsets, $R_1, R_2, \ldots, R_{n_r}$. We then let

$$p_\ell = \text{Prob}\{V \in R_\ell\} \qquad \text{for } \ell = 1, \ldots, n_r .$$

For each of the random cash-flow elements (except those which occur in the first time period), we require the determination of probability distributions conditioned on the occurrence of the various subsets, R_ℓ. The final step is to generate a $\Delta^{(\ell)}$, corresponding to each subset R_ℓ, to indicate the appropriate decisions if $V \in R_\ell$; and to specify a single δ which indicates the unconditional first stage decisions.

Therefore, the two-stage problem is to

$$\text{maximize} \quad E\{P(\delta,\Delta)\} = \sum_{j=1}^{m} E(P_j)\delta_j + \sum_{\ell=1}^{n_r} p_\ell \sum_{j=1}^{m'} E(P_j^{(\ell)})\,\Delta_j^{(\ell)} ,$$

$$\text{subject to} \quad \text{Prob}\{(\text{NCF})_1 \geq L_1\} \geq \alpha_1 ,$$

$$\text{Prob}\{(\text{NCF})_k^{(\ell)} \geq L_k\} \geq \alpha_k , \qquad \text{for } k = 2, \ldots, n; \ \ell = 1, \ldots, n_r,$$

$$\text{Prob}\{(\text{CNCF})_k^{(\ell)} \geq C_k\} \geq \eta_k, \qquad \text{for } k = 2, \ldots, n; \; \ell = 1, \ldots, n_r,$$

$$(\delta, \Delta^{(\ell)}) \in S', \qquad \text{for } \ell = 1, \ldots, n_r,$$

$$\delta_j = 0 \text{ or } 1, \qquad \text{for } j = 1, \ldots, m,$$

$$\Delta_j^{(\ell)} = 0 \text{ or } 1, \qquad \text{for } j = 1, \ldots, m'; \; \ell = 1, \ldots, n_r,$$

where $(\text{NCF})_k^{(\ell)}$ and $(\text{CNCF})_k^{(\ell)}$ are the kth-period net cash flow and cumulative net cash flow, respectively, corresponding to the event $V \in R_\ell$.

As was noted elsewhere[10], a prime consideration in the partitioning of the outcome set R is that the points within a given subset should be as similar as possible in their effect on the second stage decisions, but the effects of points in different subsets should be as dissimilar as possible. These general guidelines are sufficient to illustrate the model formulation, but obviously the actual construction of such a partition may be a very difficult task. It is paradoxical that the second-stage effects of the various points in R and, consequently, the necessary number and sizes of its partition subsets would only be known with certainty if the model were solved with each of these points considered a different subset. However, even assuming we were able to accomplish this, we would then have no need to undertake the partitioning! Obviously, no problem of practical size with a large number of elements in R (perhaps even uncountably infinite) can be solved

[10] Hillier [1969], p. 77.

point-by-point with a reasonable amount of computational effort. It therefore becomes incumbent upon the analyst to rely upon the judgement and intuition of management in determining the assignment of points (or regions) to subsets, and the number of these subsets must be limited by considerations of computational requirements.

Although the restriction on the number of subsets may lead to a relatively coarse partitioning of the outcome set, and there may be difficulties in deciding in which subsets the various members of that set belong, this procedure should nevertheless provide much more guidance for future decisions than would a more common single-stage analysis. Should sufficient information about future investment opportunities and possible outcomes be available, and should computational requirements be reasonable, the formulation can be extended in a direct way to three or more stages.

Up to this point, we have paid no attention to nonlinearity in the deterministic equivalents of the chance constraints on cash flow. (A brief discussion of these equivalents was presented in Section 1.4.) In the absence of simplifying approximation techniques, the computational complexity of nonlinear programming methods would quickly render all but the smallest multi-stage problems too time-consuming for solution. Fortunately, Hillier [1967] has enjoyed significant success in developing both uniformly tighter and uniformly looser linear approximations to chance constraints under relatively nonrestrictive assumptions, and Seppälä [1971] has developed alternative uniformly tighter linear approximations for a special class of chance constraints. A thorough discussion of these approximation techniques and the associated

solution procedures is beyond the scope of this presentation. Suffice it to say that the general procedure is to solve two different integer linear programs (one with uniformly looser constraints, and the other with uniformly tighter constraints), and then to attempt a reconciliation of any differences between the solutions while requiring feasibility with respect to the original nonlinear problem.[11]

2.3. Extension for Uncertain Future Opportunities

With few exceptions[12,13], programming models for capital budgeting have been based on the assumption that the decision-maker is faced with a group of projects which are currently available or which definitely will become available for selection at a specified time in the future. Under this assumption the single-stage formulations require that all decisions regarding current and future opportunities be made immediately, while multi-stage formulations produce decisions on current opportunities and contingency plans for the selection of the remaining ones. The inaccuracies inherent in predicting the characteristics and the availability of future investment opportunities have led many writers to incorporate additional, rather artificial, constraints on the firm's future financial posture. These constraints were used as a means of guarding against the possibility of being unable to undertake highly-desirable (currently unknown) future projects because of an excessive commitment of funds to current ones.

[11] For additional discussion and clarification, see Hillier [1967], pp. 48-51.

[12] See Fisher [1930] and Weingartner [1966a], for example.

[13] Hillier's "steady state" model ([1969], pp. 72-74) also qualifies.

The chance constraints discussed previously can be adjusted to help assure "adequate"(in management's estimation) liquidity; and the "portfolio payback" constraints and "horizon posture" constraints of Byrne, et al. [1971a,b] may be viewed as additional examples of this type of approach. However, it would seem more desirable to directly consider any information which is available on possible future opportunities, rather than to rely completely on such intuitive restrictions.

Management may very well find itself in a position of possessing significant, yet incomplete, knowledge of specific projects (or types of projects) on which decisions may be required in the future. This information may relate to whether or not the opportunities actually will arise, when they may be available, what levels of participation may be offered to the firm, which of several cash flow patterns may apply (stochastic or deterministic), or any combination of these and other factors.

The question of availability or non-availability certainly applies to research and development projects whose success may depend on the solution of specific technical problems or the development of an entirely new technology. This question may also arise if the firm relies on decisions of other organizations to request either that firm's services or those of a competitor. Obviously, in such a situation, management may not be able to determine in advance that it will be the first choice of a potential customer. Finally, there may be some possibility that the firm will not be allowed to undertake a particular project -- objections from its shareholders, restraining

orders which result from competitors' litigation, or anti-trust proceedings by the government may delay or prevent investment.

The timing of project availabilities can also be affected by influences similar to those discussed in the previous paragraph. In fact, since non-availability may be viewed as an infinite delay, any factors which may have an impact on one may also affect the other.

Participation levels may be determined by the availability of raw materials used in the firm's product, the levels of participation desired by partners in a joint venture, the amount of services or products needed by a customer, the ability of the firm to successfully fulfill a current contractual obligation, diversification requirements levied by higher authority, or any number of additional factors.

Finally, cash flow patterns may be influenced by such things as whether or not a competitor enters the field, the imposition of wage or price controls by government, the time required to successfully market a product, repair or replacement costs for vital machinery, reception of products by the public, etc.

It is apparent that few analyses could successfully consider all the possible combinations of events which might influence the characteristics and availabilities of future projects; in fact, restrictions on model development time and the degree of computational complexity might limit consideration to those few possibilities which appear a priori to be most significant. However, even the incorporation of a limited amount of the type of information discussed above can allow the model to more realistically guard against the possibility of lost future opportunities. It may still prove necessary

to employ liquidity, payback, or posture constraints to allow for the randomness of cash flow elements and the information which could not be considered by the model, but our proposed approach should allow less severe restrictions to be imposed.

We will now present our proposed method for incorporating incomplete information into a multi-stage model in the context of Hillier's "basic two-stage model", which was discussed in the previous section. A discussion of the special structure which is characteristic of both the basic model and our extension, and a brief presentation of a new integer programming decomposition algorithm for exploitation of this structure, will follow in the next chapter.

For the moment, we will consider only one subset, R_ℓ, from among those into which the outcome set (R) is partitioned in the basic model. We assume that management has designated a total of h significant possible combinations of future projects which it desires to consider in the analysis. That is, given a total of N individual projects which management believes may become available to it in the future, there are 2^N potential combinations of these which may arise. Some of these combinations may be excluded from the analysis because they contain projects which cannot possibly become available together, or because they fail to include certain projects upon whose availability that of some included projects is actually contingent. Other combinations may be excluded because their probabilities of occurrence are extremely small, or simply because giving explicit consideration to them would make the analysis too complex.

Obviously, the decision of which potential combinations to incorporate in the problem formulation must be based on the experienced judgment of both managers and analysts.

We further assume that probabilities q_i $(i = 1, \ldots, h)$ have been assigned to the events that the chosen combinations become available. Instead of determining a decision vector $(\Delta_1^{(\ell)}, \ldots, \Delta_{m'}^{(\ell)})$ as in the basic model, we will seek to determine a set of h such vectors, $(\Delta_1^{(\ell,i)}, \ldots, \Delta_{m'}^{(\ell,i)})$ for $i = 1, \ldots, h$, indicating the optimal second-stage decisions which should be made in the event that $V \in R_\ell$ and the ith combination becomes available. In order to accomplish this, we modify the objective function to include the additional decision vectors and the probabilities associated with the various combinations of projects; we also replace each of the $2n$ constraints which are indexed with ℓ by h constraints indexed by (ℓ,i) for $i = 1, \ldots, h$.

With these changes, the problem is to

$$\text{maximize} \quad E\{P(\delta,\Delta)\} = \sum_{j=1}^{m} E(P_j)\delta_j + \sum_{\ell=1}^{n_r} p_\ell \sum_{i=1}^{h} q_i \sum_{j=1}^{m'(i)} E(P_j^{(\ell,i)})\Delta_j^{(\ell,i)},$$

$$\text{subject to} \quad \text{Prob}\{(\text{NCF})_1 \geq L_1\} \geq \alpha_1 ,$$

$$\text{Prob}\{(\text{NCF})_k^{(\ell,i)} \geq L_k\} \geq \alpha_k , \qquad \text{for } k = 2, \ldots, n; \ \ell = 1, \ldots, n_r; \ i = 1, \ldots, h,$$

$$\text{Prob}\{(\text{CNCF})_k^{(\ell,i)} \geq C_k\} \geq \eta_k , \qquad \text{for } k = 2, \ldots, n; \ \ell = 1, \ldots, n_r; \ i = 1, \ldots, h,$$

$$(\delta, \Delta^{(\ell,i)}) \in S_i'' , \qquad \text{for } \ell = 1, \ldots, n_r; \; i = 1, \ldots, h,$$

$$\delta_j = 0 \text{ or } 1 , \qquad \text{for } j = 1, \ldots, m,$$

$$\Delta_j^{(\ell,i)} = 0 \text{ or } 1, \qquad \text{for } j = 1, \ldots, m'(i); \; \ell = 1, \ldots, n_r; \; i = 1, \ldots, h,$$

where S_i'' represents the feasible region for (δ,Δ) (exclusive of the chance constraints and integer requirements), given that the ith combination of second-stage projects becomes available. The model can be further generalized by making L_k, C_k, α_k, and η_k functions of ℓ and i, and by making S_i'' a function of ℓ.

CHAPTER 3

STRUCTURE AND SOLUTION OF THE MODEL

3.1. Size and Structural Aspects

Assuming that all cash flows are deterministic or that the nonlinear chance constraints are replaced with appropriate linear approximations (as discussed in Section 2.2), our multi-stage model is an integer linear programming problem in binary variables. The number of these variables is an increasing function not only of the number of projects available at each stage, but also of the number of partitions of the outcome set and the number of possible combinations of project availabilities at each stage beyond the first. In fact, using the notation of the previous chapter, the total is

$$N = m + n_r \sum_{i=1}^{h} m'(i)$$

for the two-stage model, which makes it obvious that the partitioning of the outcome set and the number of possible second stage combinations considered can greatly influence the model's complexity. With a three-stage model, each of the $n_r \cdot h$ second stage decision vectors would be replaced by as many vectors as were necessary to account for the possible second stage outcomes and the various combinations of third stage projects with might arise. So the number of stages is also a critical factor in determining the total size of the problem's decision vector.

The size of the constraint set depends on all of the above factors as well as on the length of the planning horizon (T) and the number of constraints necessary to define each of the regions S_i''. Assuming that each S_i'' requires a total of s constraints (this number may actually vary with i), and excluding the integer restrictions, the number of rows in the constraint set is

$$M = n_r(h)(s) + 1 + 2(T-1)(n_r)(h)$$

for the two-stage model. This quantity is also highly-dependent on the number of stages in the model.

When using enumerative-type zero-one integer programming algorithms, the solution effort required for most classes of problems tends to be more sensitive to increases in the number of variables than increases in constraint set sizes (excluding the integer constraints). This is due to the fact that the number of binary solution vectors, both feasible and infeasible, is an exponential function of the number of variables (n variables result in 2^n possible combinations which must be checked, implicitly or explicitly). If linear programming is used as a fathoming[14] device, the effort required by the simplex method generally increases as a polynomial function (approximately cubic) of the number of constraints, but adding more constraints will often allow the linear programs to more rapidly eliminate infeasible or unpromising integer combinations from further consideration. In

[14] See Geoffrion and Marsten [1972].

addition, increasing the number of variables in a linear program tends to increase its solution time.

Considering the potentially-exponential relationship between solution effort and the number of variables, it is clear that a method for decomposing a large integer programming problem into several smaller ones which can be solved independently could provide significant advantages. The large number of variables which result in the multi-stage model from partitioning the outcome sets at each stage and considering various combinations of project availabilities makes such a decomposition approach particularly appealing. Fortunately, the structure of the multi-stage model is readily amenable to decomposition.

Consider the block diagram of a three-stage problem shown in Figure 1. Blocks A_0, A_1, and A_2 contain the constraint coefficients for the first stage variables, blocks B_1 through B_6 contain those for the second stage variables, and the remaining blocks (C_1 through C_6) contain the constraint coefficients for the third stage variables. For purposes of exposition, we have assumed that there exist two possible outcomes at the end of the first stage, and therefore have two groups of second stage blocks (B_1-B_3 and B_4-B_6) to represent the effects of these outcomes on the second stage projects. These two outcomes can represent either the preliminary results from projects undertaken in stage one, or two different sets of second stage projects which may become available for selection. Corresponding to each of these two first stage outcomes, we have indicated three possible second stage outcomes (this number was again chosen arbitrarily for purposes of exposition) whose effects are reflected in the "C" blocks.

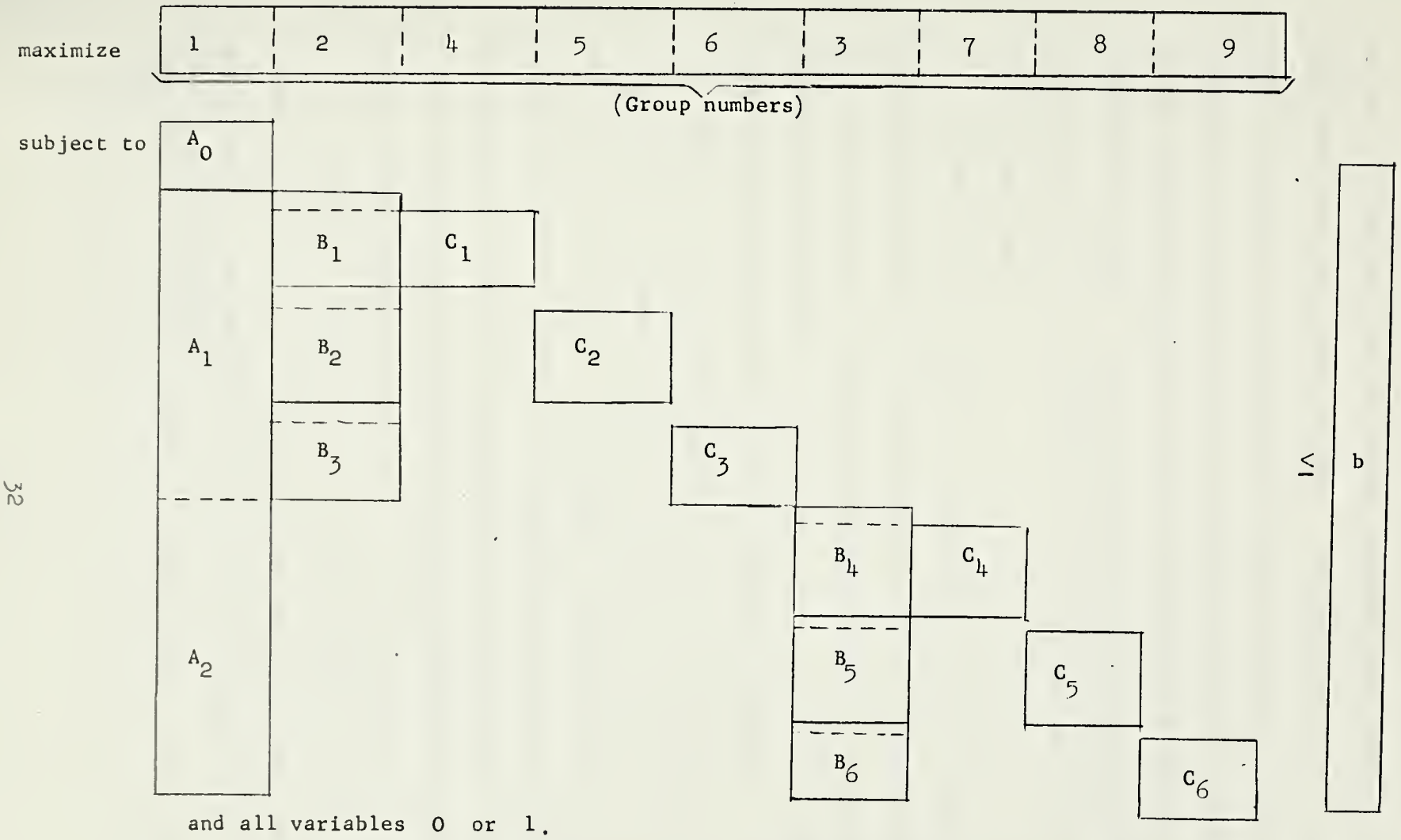


Figure 1: Block Diagram of a Three-Stage Capital Budgeting Problem

The block labeled A_0 represents the net cash flow constraint(s) for the first period, plus the mutual exclusion and contingency relationships among the first stage variables. The upper portions of each "B" block (along with the corresponding sections of blocks A_1 and A_2) represent the net cash flow and cumulative net cash flow constraint(s) for the second period as well as the interdependencies among first and second stage variables. The "C" blocks are similarly associated with third period cash flow constraints and relationships among projects of all three stages.

Assuming that n projects are available at each stage, we need $6n$ decision variables for the third stage, $2n$ decision variables for the second stage, and n decision variables for the first stage. Therefore, with a total of $9n$ variables, there are 2^{9n} possible solutions which must be implicitly or explicitly examined.

Implicit enumeration algorithms search for the optimum solution(s) to a problem by successively fixing variables at either zero or one. After each such assignment, they attempt to determine whether or not fixing the remaining variables at their most favorable (feasible) zero-one values can result in an improvement over the best known solution.[15] If the current assignment appears promising, the process continues by fixing another variable at zero or one. If the current one is ruled out, it is changed; commonly, this is done by finding the most recently-assigned variable whose alternate value has not been tried, setting that variable at its alternate value, and then disregarding all assignments

[15] See Geoffrion and Marsten [1972] for a discussion of the techniques used.

which were made after the one just changed.[16] The process is non-repeating, and terminates when no untried alternatives exist for the variables in the assignment list.

By imposing a priority ordering on the assignment process discussed above, a significant reduction can be achieved in the number of enumeration-tree nodes for our multi-stage model.[17] Specifically, we will divide the model's variables into groups, with all variables in a given group representing decisions at the same stage. It will then be demonstrated that assigning appropriate priorities to the groups and requiring that all variables in a given group be fixed before any assignments are made in a lower-priority group, can greatly enhance the efficiency of any implicit enumeration algorithm applied to our model. Referring to the group numbers indicated in the objective block of Figure 1, it is easy to see that assigning specific values to the n variables in group one reduces the remainder of the problem to two completely independent problems of $4n$ variables each. For any given assignment of the variables in group one, each of the enumeration trees for these two problems has 2^{4n} terminal nodes. Since there are 2^n possible group one assignments, the total number of terminal nodes for the entire problem is reduced by this technique from 2^{9n} to 2^{5n+1} $[2^n(2^{4n} + 2^{4n}) = 2^{5n+1}]$.

The above approach can also be applied to the solution of the two independent subproblems which resulted from assigning group one. For example, if all of the group two variables in Figure 1 are assigned

[16] This is the LIFO (Last In-First Out) rule.

[17] Schrage [1973] has also discussed the approach which we employ, and has noted its applications to several classes of problems.

values, the first of the two subproblems is further decomposed into three independent subproblems corresponding to groups four, five, and six. With 2^n different specifications of the group two variables, and 2^n possible specifications for each of the other three independent groups, we have a new total of $3 \cdot 2^{2n}$ terminal nodes to consider instead of th previous 2^{4n}. Applying the same procedure to the second subproblem, we finally reduce the total number of terminal nodes for the entire problem from 2^{9n} to $6 \cdot 2^{3n}$, $[2^n(3 \cdot 2^{2n} + 3 \cdot 2^{2n}) = 6 \cdot 2^{3n}]$. The significance of this reduction is made even more apparent by considering specific values of n -- with $n = 3$, the ratio of $6 \cdot 2^{3n}$ to 2^{9n} is approximately 1 : 43, 690. Increasing the value of n to 5 results in a ratio of approximately 1 : 178, 956, 970. This comparison makes it quite obvious that the decomposition procedure can greatly reduce the required solution times for problems derived from the multi-stage model, and thereby allow the solution of large, realistic problems which might otherwise be unassailable.

The author has developed additional techniques for reducing the computational effort required by problems of the type discussed above, and has implemented them in a decomposition algorithm for zero-one problems. A complete discussion of this algorithm and the experimental results obtained with it will be presented in Chapters 4 through 6.

3.2. Inclusion of Borrowing/Lending and Fractional Investments

As we discussed in Chapter 1, capital budgeting problems may well include opportunities for borrowing and lending funds and consider

projects which may be undertaken at any level from minor to full commitment. While it is possible to approximate such situations with sets of zero-one integer variables (each variable representing a given fraction of its associated investment opportunity), proceeding in such a manner can greatly increase the number of integer decision variables and the solution effort required for the problem (as discussed in the preceding section). It is normally preferable, therefore, to represent such decisions by continuous variables.

Unfortunately, the solution procedure which we outlined above runs into some serious difficulties when an attempt is made to allow continuous decision variables in the model. With all decision variables restricted to zero or one, the value-assignment alternatives for our enumerative procedure are quite clear; but when we allow one or more variables to have an infinite number of possible values, and when the values of these variables will have effects beyond the stage at which they must be fixed (for decomposition), the question of where to set these variables becomes rather difficult. To illustrate this point, we refer to Figure 2 below. Figure 2 is a block diagram of a two-stage problem with two second-stage possibilities (subproblems), and indicates the variables and matrix entries associated with the blocks. The decomposition procedure requires us to specify values for variables x_1, x_2, and x_3, and then to address separately the two independent subproblems which result. Assume that x_1 and x_2 are zero-one variables for which we have chosen specific values, and that x_3 is a continuous variable such that $0 \leq x_3 \leq 1$. To decompose the problem, we need to choose the best setting of x_3 relative to the

maximize $\quad E(P_1) + E(P_2) + E(P_3) + \sum_{i=4}^{9} E(P_i)$

subject to

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	
a_{11}	a_{12}	a_{13}	x_4	x_5	x_6				$\leq b_1$
a_{21}	a_{22}	a_{23}	b_{21}	b_{22}	b_{23}				$\leq b_2$
a_{31}	a_{32}	a_{33}	b_{31}	b_{32}	b_{33}	x_7	x_8	x_9	$\leq b_3$
a_{41}	a_{42}	a_{43}				b_{41}	b_{42}	b_{43}	$\leq b_4$
a_{51}	a_{52}	a_{53}				b_{51}	b_{52}	b_{53}	$\leq b_5$

Figure 2: Two-Stage Problem

assigned values for x_1 and x_2. The exact determination of this best setting under most circumstances requires the solution of a mixed-integer programming problem involving x_3 and the second-stage variables -- exactly the problem we are trying to avoid through decomposition.

Fortunately, there are several cases in which the proper choice of x_3 is immediately obvious, even without solving the mixed-integer program mentioned above:[18]

I. If $E(P_3) \geq 0$, $a_{13} > 0$, $(a_{23}, a_{33}, a_{43}, a_{53}) \leq 0$, and $a_{11}x_1 + a_{12}x_2 \leq b_1$, then

$$x_3 = \min\left[1, \frac{b_1 - a_{11}x_1 - a_{12}x_2}{a_{13}}\right] .$$

Since the investment represented by x_3 makes a non-negative contribution to total profit and does not require the use of

[18] Schrage [1973], p. 473, refers to such possibilities in general terms, but provides no specific examples.

resources which may be needed by other projects, there is no reason not to accept it at the maximum level permitted by the first constraint.

II. If $E(P_3) \geq 0$, $a_{13} \leq 0$, and $(a_{23}, a_{33}, a_{43}, a_{53}) \leq 0$, then $x_3 = 1$. Since the investment gives a non-negative profit and has a non-positive consumption of all resources, there is no question of its desirability.

III. If $E(P_3) \leq 0$, $a_{13} < 0$, $(a_{23}, a_{33}, a_{43}, a_{53}) \geq 0$, and $a_{11}x_1 + a_{12}x_2 > b_1$, then $x_3 = [b_1 - a_{11}x_1 - a_{12}x_2]/a_{13}$. In this case, the investment is undesirable from the points of view of profit and consumption of resources in the last four constraints. However, given the assigned values for x_1 and x_2, the first constraint can only be satisfied by accepting the investment at a level greater than or equal to that indicated. Since there can be no advantage in accepting at a greater level, the indicated one is best.

IV. If $E(P_3) \leq 0$, $a_{13} \geq 0$, and $(a_{23}, a_{33}, a_{43}, a_{53}) \geq 0$ then $x_3 = 0$. Since the investment makes a non-positive contribution to profit and requires a non-negative consumption of resources in all constraints, it is clearly undesirable.

Cases I and III are certainly the most interesting of the ones presented, since the other two describe investments which should be excluded from the analysis before it reaches the computational stage.

Two very common types of "investments" which are included in cases I and III, respectively, are contracts for lending or borrowing funds (x_3 indicates the amount transacted). For example, if the borrowing rate equals or exceeds the discount rate used to calculate present value, then borrowing is an unprofitable activity with $E(P_3) \leq 0$. Assuming, for instance, that the first, second, and fourth constraints of Figure 2 are on net cash flow, and that the first of these applies to the first time period while the other two apply to the second period, we have the following situation: $a_{13} < 0$, $a_{23} > 0$, and $a_{43} > 0$. This indicates that the money borrowed in the first period must be repaid in the second, with interest. If the third and fifth constraints are on cumulative net cash flow through the second period (one such constraint for each possible first-period outcome), we have $a_{33} > 0$ and $a_{53} > 0$ because more money is repaid than was borrowed. Had our model included constraints on more periods, the additional net cash flow constraints would have had zero entries while the cumulative net cash flow constraints would have had positive ones. This meets the requirements for case III. Similar reasoning places lending activities in the category of case I.

The results for cases I and III can easily be extended to allow for situations in which the applicable rates for lending and borrowing are decreasing and increasing functions, respectively, of the amounts involved. This is accomplished by having different decision variables represent the amounts contracted under the various rates. Since the decision variables in our model are bounded above by one, we would

indicate the maximum amount available under a given rate by the value of the associated variable's coefficient in the constraint applying to the period in which the borrowing or lending will take place.[19] The coefficients in the constraints for later periods are, of course, adjusted appropriately. The obvious solution then is to make commitments under the most attractive rates first, proceeding as far as necessary (in decreasing order of attractiveness) to contract for the desired amount.[20]

There is one circumstance in which continuous-variable projects can be handled by the decomposition procedure, even if they do not fit in special categories such as the ones we have described. This occurs when the projects are associated with blocks which represent subproblems at the lowest level of decomposition. In this situation, we can specify values for all of the integer variables (and the continuous variables of types I and III which are needed for decomposition) before attempting to determine the best values for these continuous variables. Having accomplished this, we can use linear programming (an integral part of most enumerative algorithms) to optimize the resulting subproblems with respect to the continuous variables.

3.3. Summary

The capital budgeting model presented in this chapter is designed to provide a strategy approach to the selection and timing of risky interrelated investments when the availability of future opportunities

[19] Alternatively, of course, we could employ separate upper bound constraints on the variables.

[20] See Weingartner [1967], pp. 168-172 for a discussion of such results for linear programs with separable convex objective functions.

is uncertain. The model employs chance-constrained programming to express the risk preferences of the firm over time, thus avoiding the difficult problem of constructing or postulating an appropriate utility function. With either deterministic cash flows or suitable linear approximations to the chance constraints, the problem reduces to one of multi-stage integer linear programming under uncertainty. The solution of this multi-stage problem can be accomplished using a decomposition approach which allows for the inclusion of borrowing, lending, and certain other types of non-integer activities.

CHAPTER 4

THE DECOMPOSITION METHOD: GENERAL DISCUSSION

4.1. Applications of the Method

The algorithmic procedure to be described in this chapter is a refinement of the technique discussed in Chapter 3 and also presented in Schrage [1973]. Although we have confined our discussion up to this point to the development of an improved multi-stage model for capital budgeting, the proposed solution procedure can be applied to all pure-integer (and certain mixed-integer) programs with dual-angular constraint matrices. The class of such problems includes not only the multi-stage problems discussed earlier, but also (with appropriate column rearrangement) those with structures similar to that of Figure 3. Such structures often arise in problems involving interactions among several time periods, operating locations, or steps

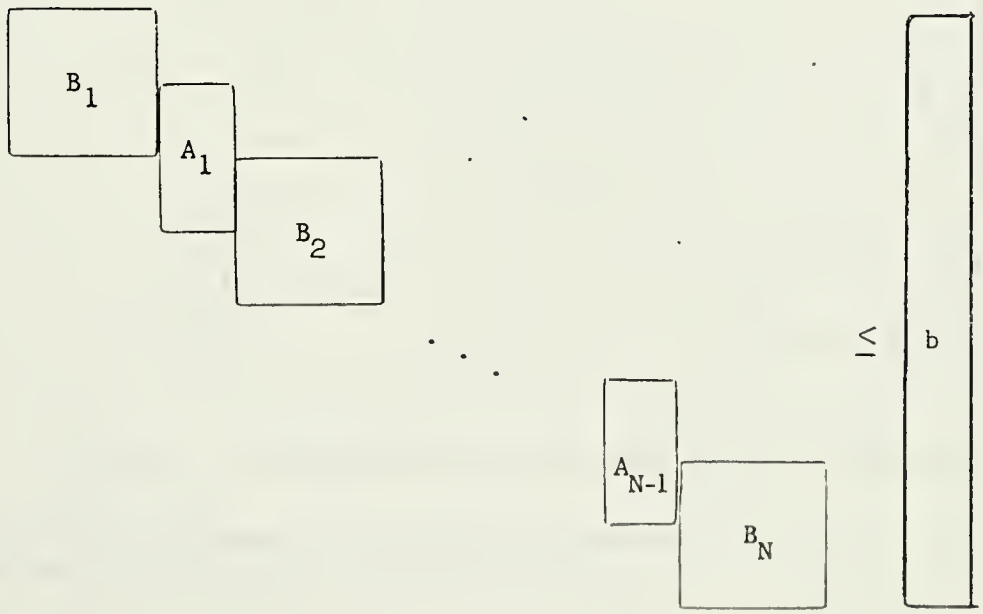


Figure 3: Example of a Dual-Angular Structure

of manufacturing processes.[21] Quite commonly, the linking columns (represented by blocks A_1 through A_{N-1} in Figure 3) reflect inventories of intermediate or finished products which may be transferred among activities or stored for future use. In such cases, these linking blocks may actually tie together more than two "production" activities (as indicated by blocks B_1 through B_N in Figure 3) to ensure that inventory balance constraints are properly adjusted for transfers of materials and products. In any case, by grouping together these linking blocks, as shown in Figure 4, we can obtain an equivalent formulation having the same structure as the multi-stage capital budgeting problems of previous chapters. In this example (as was also noted and

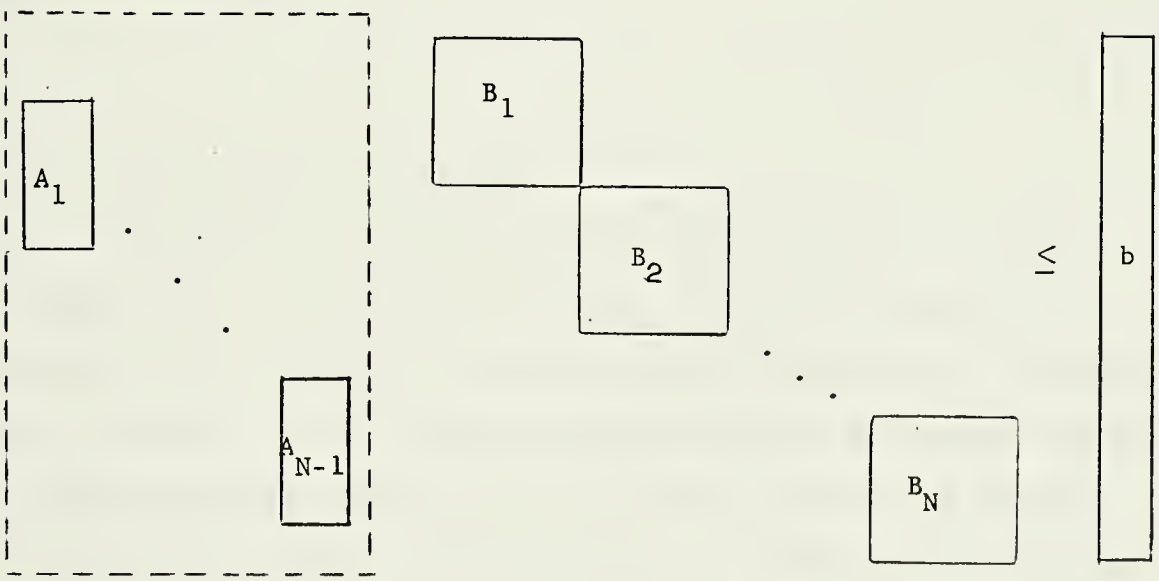


Figure 4: Dual-Angular Structure Equivalent to Figure 3

[21] Driebeek [1969], Chapter 5, contains an excellent discussion of situations in which such interactions arise.

discussed in Schrage [1973]) the linking block itself possesses a special structure which may be exploited to further reduce the number of possible solutions to be considered. The general dual-angular structure, however, allows any arrangement of coefficients in the linking block.

Our proposed procedure is not restricted in its application to any one of the various implicit enumeration algorithms which have been presented in the integer programming literature[22], and is equally applicable to both binary and general-integer linear programs. For this reason, we will restrict our attention in the present chapter to explaining the reasons for the development of the method and to discussing its applicability to enumerative-type integer linear programming algorithms. The details of our implementation of the method within the framework of a specific algorithm will be presented in the next chapter.

4.2. Motivation for the Method

The number of variables in a dual-angular linear or integer program grows in direct proportion to the number of diagonal submatrices in its constraint matrix (assuming an equal number of variables per submatrix). As was noted in our discussion of multi-stage capital budgeting problems (Section 3.1), the number of these submatrices increases rapidly with the combinations of possible future events and project availabilities being considered. The discussion also pointed out that the total number of enumeration-tree terminal nodes for an integer program is an exponential function of the number of its variables, and

[22] See Geoffrion and Marsten [1972] for a comprehensive survey of these methods.

outlined a procedure for reducing the number of nodes which must be explicitly considered for dual-angular integer programs. This procedure requires that a priority be placed on specifying values for those variables which, when so specified, reduce the remainder of the problem to two or more separate subproblems which can be optimized independently. The same approach can be applied to these subproblems (and their descendents) if they also possess a structure which is amenable to such decomposition.

The number of rows in a dual-angular problem's constraint matrix also increases proportionately with the number of submatrices added to it. Therefore, any algorithm whose time requirements are highly row-dependent may run into significant difficulty with such problems, even if the number of enumeration-tree nodes considered is reduced by the above decomposition procedure. After a given group of linking variables is completely specified and its associated subproblem (or main problem) is decomposed into separate subproblems, this difficulty can be alleviated by operating in only that portion of the constraint matrix which corresponds to the particular subproblem being optimized. However, if the number of linking variables is large, a great deal of work may need to be done in the full matrix before decomposition occurs.

Most successful enumerative algorithms employ linear programming as a fathoming device, as an aid in determining the sequence in which the variables are assigned, and as a means for choosing the assigned values. The amount of time required to perform pivot operations grows much faster than linearly with the number of rows in the constraint matrix.[23] In addition, auxiliary calculations required for the enumeration

[23] A cubic dependence seems to be generally accepted.

scheme, such as the derivation of penalties, may also require disproportionately larger amounts of time as the number of rows increases. Although it may be possible to restrict pivot operations and auxiliary calculations to the appropriate submatrices after a problem is decomposed, this will require additional bookkeeping and recoding of available algorithms. Even if this is accomplished in an efficient manner, the problem of additional time requirements remains up to the point of decomposition.

Our method is designed not only to greatly reduce the number of enumeration-tree nodes which must be considered for dual-angular integer programs, but also to alleviate the aforementioned problem of disproportionate row-dependence.

4.3. Description of the Method

Our decomposition approach to the solution of dual angular integer programs may be considered a synthesis of Schrage's decomposition technique[24] with a pre-solution relaxation which reduces the number of calculations required at each node of the solution tree. Rather than re-optimizing at each node a linear program corresponding to the entire constraint matrix, it is only necessary to re-optimize a number of smaller subproblems, as determined by the number of diagonal submatrices in the given problem. In addition, the complexity of auxiliary calculations required by some algorithms can be sharply reduced by this approach.

[24] See Schrage [1973].

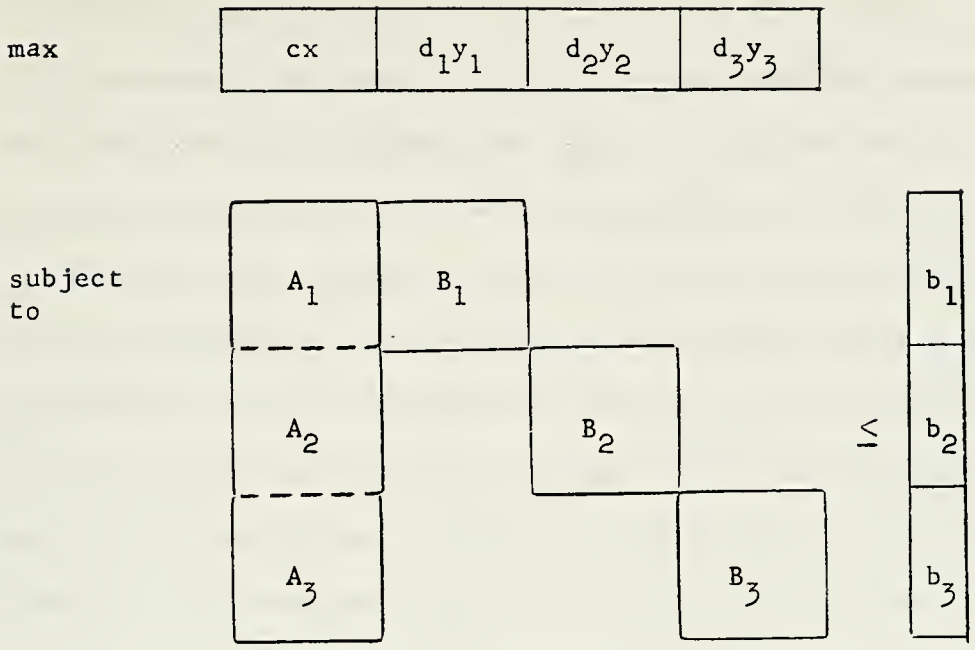


Figure 5: Block Diagram of a Two-Stage Problem

Consider the block diagram of Figure 5. Assuming one knows each subproblem's contribution to the vector c (i.e., the "allocation" of c among the subproblems), it is clear that this problem actually represents three separate integer programs, on which we have levied the additional requirement that their final solutions must agree with respect to the decision vector x. It seems quite reasonable to expect that a relaxation of this condition would cause a significant reduction in the required computational effort, provided that one can ensure that the condition will eventually be satisfied. Fortunately, this is quite easy to accomplish within the framework of implicit enumeration.

Our solution procedure specifies that the requirement on x be dropped completely from the formal statement of the problem, which immediately decomposes into three unrelated and smaller integer programs. This relaxation can be seen as a special case of Geoffrion's

[1972] Lagrangean Relaxation technique if we first write the problem in the equivalent form shown in Figure 6. The identity constraints represent the condition to be relaxed, and we can accomplish this by moving them up into the objective function with Lagrange multipliers of zero. This condition is then reimposed one variable at a time by (1) insisting that all x-variables in a subproblem be assigned values before any assignments of the y-variables are made, and (2) requiring that corresponding x-variables in the various subproblems be assigned in the same order and to the same values. In effect, then, we solve the subproblems in parallel with identical x-variable (first stage) solution trees. This obviously results in an implicit retightening of the original relaxation as successive variables are set to the same values. When all of the x-variables have been fixed, the relaxation is irrelevant, the problem decomposes of its own accord, and optimization of the independent subproblems leads to an optimal solution relative to the given assignment of first stage variables.

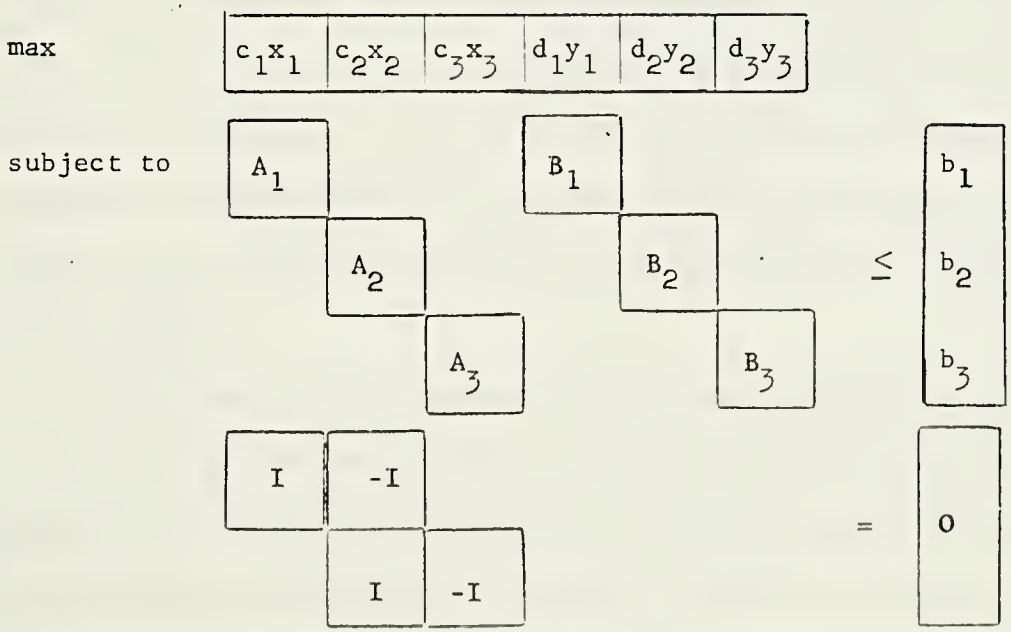


Figure 6: Equivalent Formulation of Problem in Figure 5

Of course, there is a price that must be paid for the reduction of computational complexity, and that is in our ability to fathom first stage nodes. Since the sum of the subproblem objective bounds will normally exceed the bound which could be obtained from the unrelaxed problem (it obviously cannot be less), it is quite possible that our method will require further work before a given branch of the solution tree is recognized as unpromising. However, one can expect that this disparity will be reduced as additional variables are fixed (the relaxation is tightened); and since most fathoming will tend to occur after several such branches, rather than at the beginning of the search process, the loss may well be minimal. Another mitigating factor is that a first stage node may be fathomed by infeasibility of an associated subproblem or by an excessive sum of subproblem bounds before all of the subproblems are updated to reflect the last variable assignment. If this occurs, it is unnecessary to update the remainder before backtracking in the solution tree, thus saving additional time. By using a heuristic ordering of the subproblems according to relative "tightness" or size of objective function coefficients, one can enhance the probability that fathoming will occur before all or most are updated. Information gathered during the solution process can be used to reorder the subproblems, either by the algorithm alone or interactively with an operator.

It is apparent that this procedure places no restrictions on the type of algorithm which can be used to optimize the subproblems, except that the rules used for selection of first stage branching variables and directions must be changed to utilize information

generated from the various subproblems. Most of the commonly-used rules for these purposes (e.g., "greatest penalty", "priority", or "least total infeasibility after branching") can be modified to resolve any differences in the selections indicated by the subproblems. For example, in a penalty-type algorithm, the proper branch can be determined by considering the sums of each variable's up and down penalties in the subproblems and branching away from the largest sum.

The first stage objective coefficients for a two-stage capital budgeting problem are determined by summing the coefficients corresponding to its subproblems, so the appropriate allocation of these coefficients among the subproblems of our relaxed version is obvious. In cases where no such natural division is known, one can be specified arbitrarily by the analyst. The development of guidelines for this procedure may be a fertile area for additional research.

CHAPTER 5

THE DECOMPOSITION METHOD: IMPLEMENTATION

5.1. Introduction

To evaluate the proposed method, a penalty-type implicit enumeration algorithm was developed for application to two-stage, dual-angular, binary integer programs. The algorithm employs Tomlin's improved penalties[25] for fathoming and branch-guidance, and utilizes a LIFO (Last In - First Out) search strategy throughout. The LIFO strategy was selected for ease of programming, small storage requirements, and for its support of efficient reoptimization of the linear programs involved.[26] The algorithm also requires that all first stage variables be assigned zero-one values before any such assignment of second stage variables.

As stated in the preceding chapter, relaxation into independent subproblems prior to complete specification of the first stage variables necessitates alteration of the commonly-used criteria for branch selection and fathoming. Two methods, based respectively on surrogate constraints and sums of penalty-generated subproblem bounds, were developed and tested. A full explanation of these methods is contained in this chapter, and a comparison of computational results is given in the next.

---

[25] See Tomlin [1970] and [1971].

[26] See Geoffrion and Marsten [1972] and Glover [1965] for discussions.

The reader may find it helpful to examine the simplifed flow diagram of the algorithm in Figure 7. Although many details have been omitted, the diagram illustrates the sequence of major steps and decision points in the basic algorithm. We will discuss these steps and various alternatives which may be applied to them in subsequent sections, and will refer back to this diagram for ease of exposition.

5.2 Penalties

Development of the concept of "penalties" by Driebeek [1966] has led to a considerable amount of research in this area, and to several successful integer programming algorithms.[27] The concept is quite elementary, but provides a powerful tool for estimating the objective function effects of assigning integer values to currently-unfixed variables. This information can be used to specify the order in which variables are fixed and to indicate when the current assignment of variables must be changed to possibly produce a better solution than is already known. Use of this stopping rule helps avoid the time-consuming task of reoptimizing the linear programming problem for both branches on a new variable in order to recognize that an assignment change is necessary. In addition, when one of two possible branches on a given variable will lead to an infeasible or inferior solution, penalties may indicate this fact and eliminate the need for such determination via the simplex method.

[27] See Davis, et al. [1967], Roy, et al. [1970], and Benichou, et al. [1970], for example.

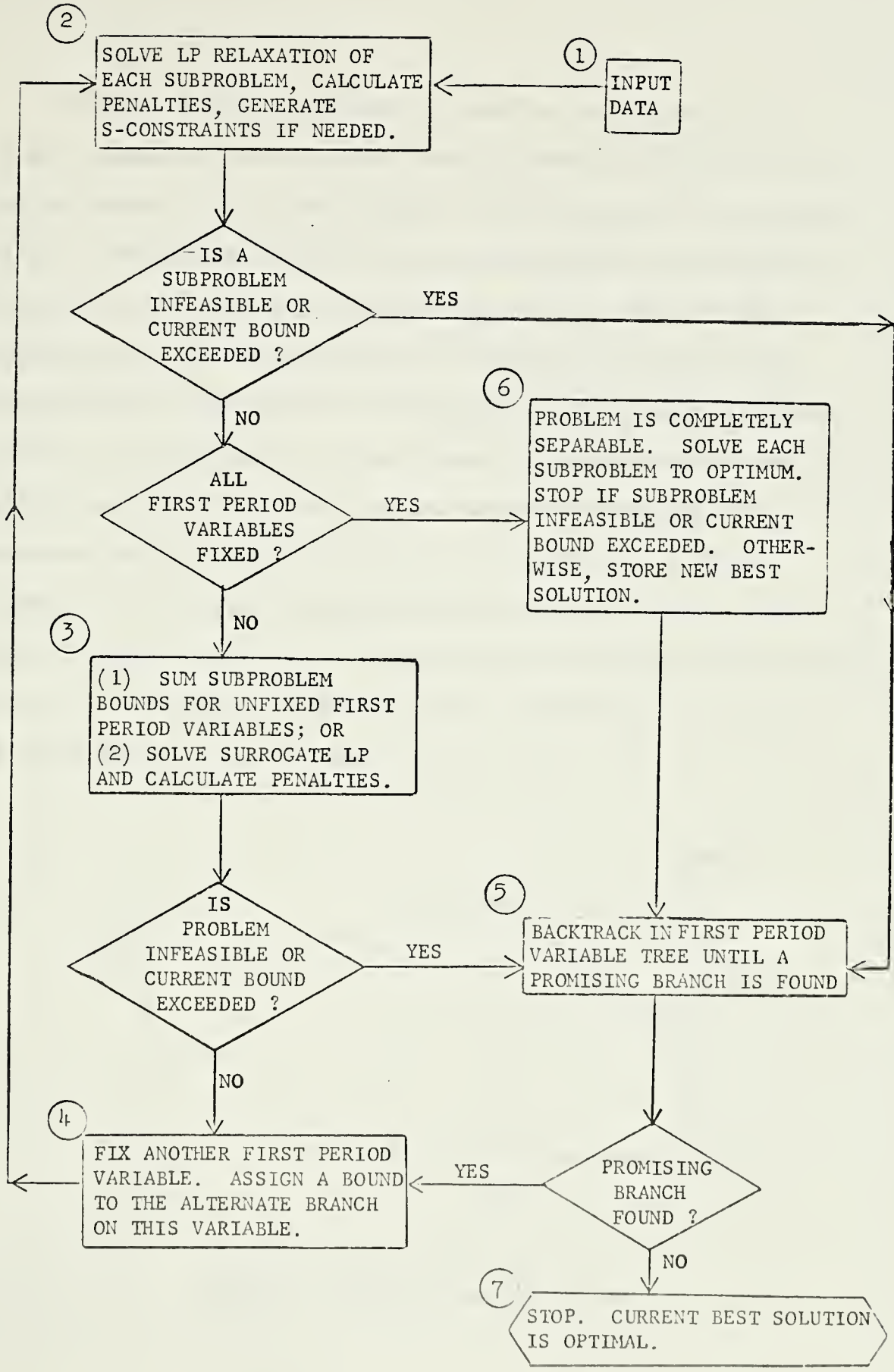


Figure 7. Simplified Flow Diagram of the Algorithm

Assuming that one is seeking to maximize the objective of a linear program, it is a relatively straightforward exercise to show that the maximum is a piecewise-linear concave function of any problem variable. The maximum of this function, as indicated in Figure 8, is attained at that value, $\bar{x}_j$, which the variable x_j takes in the optimal linear programming solution. Information contained in the optimum LP matrix can be used to calculate the slopes of line segments a and b. Multiplication of these slopes by the distance between $\bar{x}_j$ and the appropriate ordinate gives the values of the up and down penalties, designated by P_U and P_D. As is made obvious by the diagram, subtracting the appropriate penalty from the value of $\bar{z}$ gives an upper bound on the maximum objective value which can be obtained if x_j is fixed at zero or one. (In this illustration, $\bar{z} - P_U$ is the exact value of max z subject to $x_j = 1$.)

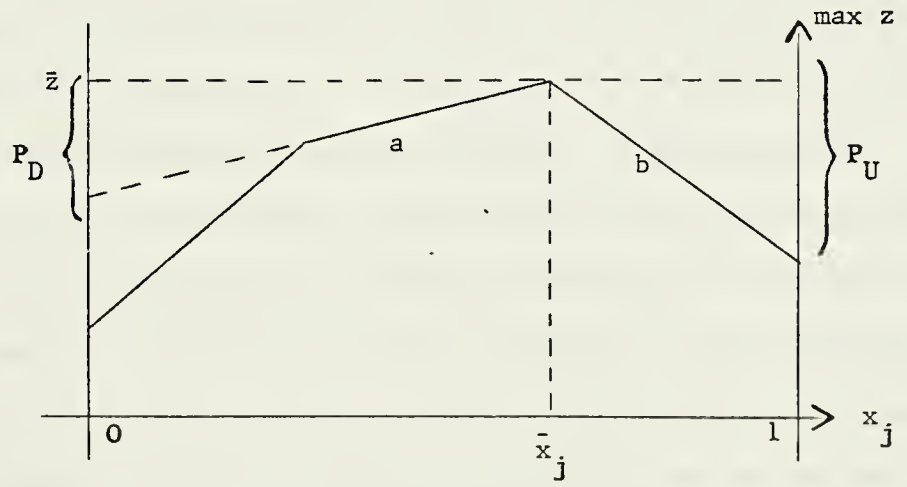


Figure 8. Graph of max z vs. x_j

By calculating the up and down bounds for all basic variables in the optimum LP solution, determining the maximum of the two bounds for each variable, and finally finding the minimum of these maxima, we obtain an upper bound on the best integer solution to the problem (relative to the current assignment of zero-one values to the fixed variables). If this bound is less than the objective value of the best known integer solution, a change in the current assignment is obviously necessary (Figure 7, Step 5). Given that the present branch is not fathomed in this manner, the various bounds can be compared with the best known solution. Any bounds which fail to exceed this value indicate that their associated variables must take on the opposite values if the current assignment is to lead to an improved solution.

Assuming that a complete assignment of the variables has not been achieved and that the current partial assignment still appears promising, it becomes necessary to choose a new branching variable and a value for that variable. A heuristic choice rule which has been successfully employed is the following: Find the minimum among all up and down bounds -- then branch in the opposite direction on the variable achieving this minimum. The objective function bound on the branch in the direction of the minimum is stored (along with a description of that branch) for future reference (Figure 7, Step 4).

Tomlin [1970, 1971] presented two methods for the derivation of stronger penalties than may be obtained via the simple procedure described above. The first of these methods utilizes the fact that

forcing a fractional variable to an integer value requires that a nonbasic variable enter the basis from either its upper or lower bound.[28] If this entering variable is one which is required to be an integer, then one can assume at least a unit change in its value for purposes of computing penalties. That is, two sets of penalties may be computed for each basic variable -- one in the manner described previously, and one which also assumes a unit change in the driving variable. One then uses the larger of the two penalties determined for each direction of change in the basic variable to calculate bounds.

Tomlin has also shown that calculation of a bound based on a Gomory cut on the fractional variable results in one that is at least as small as the larger of the two improved penalty bounds discussed above. This "Gomory bound" can therefore be used as a more powerful fathoming test than is given by the minimum-penalty bound. A method for strengthening Tomlin's improved penalties even further, but at the cost of additional computational effort, has been developed by Armstrong and Sinha [1973].

Our use of Tomlin's improved penalties in the algorithm is as follows: Given a partial assignment of the first-stage variables and an assigned value for the next branching variable, we proceed to reoptimize each independent subproblem with respect to the new branch (Figure 7, Step 2). Each revised linear program's objective value replaces the previous bound for its subproblem, and the revised sum of subproblem bounds is checked against the best known solution. If

[28] Assuming the upper-bounding technique for linear programming is being used, a variable can be nonbasic at either its lower bound (usually zero) or its upper bound.

the branch is not fathomed, penalties are calculated for all non-slack basic variables, and a new subproblem bound is determined from the largest "Gomory penalty". This bound is also summed with the other subproblem bounds and checked against the best known solution to the problem. We proceed to address each subproblem in this way until fathoming occurs or all are updated.

In the case of fathoming, the LIFO rule guides selection of the next partial assignment to be considered (Figure 7, Step 5). Otherwise, for each unfixed variable from the original problem, two quantities are determined. The first such quantity is found by summing over all subproblems the bounds given by each subproblem's objective and the down penalty associated in that subproblem with the variable of interest. If, in some subproblem, the variable is already at its lower bound, we use only the LP objective as a "down bound" from that subproblem. If the variable is nonbasic at its upper bound, we use its relative cost as a down penalty. The sum derived from this process gives a valid upper bound (assuming maximization) on the best integer solution to the given problem when all fixed variables are set at their assigned values and the variable of interest is forced to its lower bound. The second quantity is calculated in a similar manner to estimate the total effect of increasing the variable to its upper bound (relative costs are now used as up penalties in those subproblems in which the variable is nonbasic at its lower bound), (Figure 7, Step 3). The variable with the smallest quantity of either type is chosen as the next branching variable, and the branch is made in the opposite direction from that sum. (This is

analogous to the branching criterion which would be used if the problem were not relaxed into independent subproblems.) Of course, any variables whose bound sums force particular assignments are given those values and the reoptimization cycle is performed again prior to selecting a branching variable.

After values have been assigned to all first stage variables and decomposition has occurred, the algorithm proceeds to solve each subproblem to its integer optimum (relative to the first stage assignment), (Figure 7, Step 6). The only special features employed during this phase of the procedure are (1) bounds obtained for second stage branches are added to the other subproblem bounds to see if the branches are worth considering (this is in addition to checks against the individual subproblem's best known integer solution), (2) as they are obtained, subproblem integer optima replace the bounds determined at the end of the first stage, and (3) naturally-integer LP solutions are utilized as they occur. If, at any time, a subproblem is found to be infeasible or the sum of subproblem bounds falls below the required level, the remaining subproblems need not be optimized. The algorithm then returns to make a change in the first stage assignments.

Termination of the solution procedure occurs when the upper bounds of all unexplored first stage variable branches are inferior to the objective value of the best known integer solution. If no feasible integer solution has been found, the procedure may also terminate if no potentially-feasible partial assignment remains (all branches have upper bounds of minus infinity). Since there exists

a finite number of possible zero-one assignments to a problem's variables and the LIFO rule prevents repeated consideration of any such assignment, the algorithm will obviously require only a finite number of steps.

5.3. Surrogate Constraints

A considerable amount of research has been done on the development of logical tests for fathoming branches without the time-consuming use of linear programming. Work along this vein was initiated by Balas [1965] and has been extended with varying degrees of success in Glover [1965], Balas [1967], and Geoffrion [1967, 1969]. These latter studies have recognized the limitation that logical tests can be efficiently applied to only one constraint at a time, and have attempted to generate single constraints which capture some of the joint logical implications of two or more constraints considered simultaneously. Such single constraints, which serve in the place of multiple original constraints for logical testing, were dubbed "surrogate constraints" by Glover [1965]. Geoffrion [1969] has shown that the most powerful surrogate constraint is one determined by a weighted sum of the given constraints, where the weights are the optimal dual variables generated from a linear programming solution to the problem. By employing logical tests on such a constraint (and possibly some earlier ones) before and after selecting a branching variable, one may often fathom a current branch without further recourse to linear programming. Geoffrion [1969] has demonstrated that this approach results in significant reductions in the amount of computational effort required to solve a large variety of problems.

As an alternative to the sum-of-bounds approach described previously, a method for utilizing surrogate constraints for fathoming and guidance of branch selection was developed for our algorithm. Our use of surrogate constraints differs substantially from Geoffrion's [1969] in that Balas-type logical tests are not employed and the constraints are only generated during first stage branching. This should not be viewed as a rejection of such logical tests, but rather as a limitation imposed for the purpose of experimentation. The primary objective of our use of surrogate constraints was to determine whether or not they could provide an efficient means for regaining some of the information lost as a result of the relaxation to independent subproblems. Specifically, we sought to represent the set of constraints for each subproblem by a single surrogate constraint and to reimpose the requirement that first stage variables have the same values in each subproblem. The surrogate constraints are generated as byproducts of the subproblems' linear programs (using the optimal dual variables to form weighted sums of the subproblem constraints), with objective rows being assigned zero weights. (The reason behind this assignment of zero weights is that, while one can insist that the aggregate of these objectives exceed the value of the best known integer solution, the value of any given subproblem's contribution to this sum is immaterial.)

A linear program is formed from the aggregate objective and the subproblem surrogate constraints, with corresponding first stage variables in the constraints and objective made identical once again

(Figure 7, Step 3). Solution of this surrogate linear program leads to a valid upper bound on the best integer solution which can be obtained from the current branch, thereby providing a supplement to the sum-of-bounds fathoming test described in the preceding section. Assuming that fathoming does not result immediately, penalties are calculated for the non-slack basic variables of the surrogate LP, and a revised bound is determined from the largest Gomory penalty. If fathoming still does not occur, the penalties are used to check for inadmissible values of unfixed first stage variables and to choose the next first stage branch. Once a sufficient number of iterations are performed to assign values to all first stage variables, a surrogate linear program can produce no additional information, and the naturally-decomposed problem is solved by independent subproblems as discussed previously. Had we chosen to supplement the solution procedure with logical tests, we would continue to generate surrogate constraints and to perform the appropriate logical tests during the post-decomposition solution of each subproblem.

As is discussed in the chapter on computational experience, the use of surrogate linear programs proved to be more time-consuming than the basic sum-of-bounds approach for fathoming and branching. This may be at least partially attributed to the fact that the surrogate linear programs were used to supplement, rather than to replace, the penalties calculated in each subproblem. However, we conjecture that the time required to set up and solve these surrogate problems will generally exceed that which can be saved by suppression of subproblem penalty calculations.

5.4. Steps of the Algorithm

The preceding discussion has explained the logic and rationale behind our solution procedure within the framework of the simplified flow diagram of Figure 7. While this presentation serves to locate and outline the most important steps of the algorithm, a more detailed discussion is necessary to adequately explain the sequence and interrelationship of the various steps and to exhibit the more intricate mechanisms of the procedure. In addition, the following exposition will enable us to show clearly the alternative uses of penalties and surrogate constraints, and to point out some specific modifications which may be made in future efforts to improve the performance of the algorithm.

0. We are seeking to solve the following zero-one integer linear program:

$$\text{maximize} \quad \sum_{\ell=1}^{L} \sum_{j=1}^{n_1} c_{\ell j} x_j + \sum_{\ell=1}^{L} \sum_{j=1}^{n_{2\ell}} d_{\ell j} y_{\ell j} ,$$

$$\text{subject to} \quad (1) \quad \sum_{j=1}^{n_1} a_{0ij} x_j \leq B_{0i} , \qquad \text{for } i = 1, \ldots , I_0 ,$$

$$(2) \quad \sum_{j=1}^{n_1} a_{\ell ij} x_j + \sum_{j=1}^{n_{2\ell}} b_{\ell ij} y_{\ell j} \leq B_{\ell i} ,$$

$$\text{for } \ell = 1, \ldots , L; \; i = 1, \ldots, I_\ell ,$$

$$(3) \quad 0 \leq x_j \leq 1 \qquad \text{for } j = 1, \ldots , n_1 ,$$

$$(4) \quad 0 \leq y_{\ell j} \leq 1 \qquad \text{for } \ell = 1, \ldots , L; \; j = 1, \ldots, n_{2\ell} ,$$

(5) all variables integer ,

where x and y represent first and second stage variables, respectively. We also have

L = number of subproblems (diagonal blocks),

n_1 = number of first stage variables,

$n_{2\ell}$ = number of second stage variables in the ℓth subproblem,

$c_{\ell j}$ = contribution to profit from subproblem ℓ when first stage project j is undertaken,

$d_{\ell j}$ = profit from undertaking second stage project j in subproblem ℓ,

$a_{\ell ij}$ = use of ith resource by first stage project j in subproblem ℓ,

$b_{\ell ij}$ = use of ith resource by second stage project j in subproblem ℓ,

I_0 = number of resource constraints applying solely to first stage variables ("subproblem zero"),

I_ℓ = number of resource constraints in subproblem ℓ ($\ell = 1, \ldots, L$),

and

$B_{\ell i}$ = resource limitation of ith constraint in subproblem ℓ ($\ell = 0, \ldots, L$).

1. Relax the problem into independent subproblems by replacing each variable x_j with $x_{\ell j}$, where $\ell = 1, \ldots, L$, and appending constraints (1) to each such subproblem (with appropriate subscripting of x). Further relax the problem by dropping the integer requirements given by (5).

2. Initialize the solution procedure by choosing an ordered set, S, (usually empty) of first stage variables to be fixed, and assigning values of zero or one to these variables. If an integer solution to the problem is known a priori, set z^* equal to its objective value and store the solution as the current best. Otherwise, set $z^* = -\infty$. Go to step 4.

3. Choose a new first stage branching variable according to one of the following two procedures:

a. Sum-of-Bounds: Consider, for each variable not contained in S, the sum of its up subproblem bounds and the sum of its down subproblem bounds as determined in step 6. Select that variable x_j with the smallest sum of either type, add the variable to the ordered set S, and assign to all variables $x_{\ell j}$ ($\ell = 1, \ldots, L$) the 0-1 value of x_j which is opposite to the smallest sum. Note the bound on the alternate branch on x_j.

b. Surrogate LP Penalties: Among all first stage basic variables in the last surrogate linear program, find the one with the largest up or down penalty, $\hat{P}$. Add this variable, x_j, to the set S and assign to all variables $x_{\ell j}$ ($\ell = 1, \ldots, L$) the 0-1 value of

x_j which is opposite to this largest penalty. The bound associated with the alternate branch on x_j is $\hat{z}$ minus the larger of $\hat{P}$ and $\hat{P}_G$.

If all unfixed first stage variables are nonbasic in the last surrogate LP, select the one with the largest relative cost. Add this variable, x_j, to S and assign to all variables $x_{\ell j}$ ($\ell = 1, \ldots, L$) the present value of x_j in the surrogate LP. The alternate bound on x_j is $\hat{z}$ minus its relative cost.

4. Select a subproblem to update with respect to the last variable assignment. Reoptimize this subproblem as a linear program while ensuring that all variables corresponding to members of S are nonbasic at their upper or lower bounds (one or zero), as appropriate. If the subproblem is now infeasible, go to step 10.

5. Replace the previous upper bound on an integer solution to this subproblem by the updated LP objective $\bar{z}$ (if smaller). If the sum of all subproblem bounds is now less that z^*, go to step 10.

6. Calculate penalties for all unfixed first and second stage variables and determine the largest Gomory penalty, P_G^*. Replace the bound for this subproblem by $\bar{z} - P_G^*$. If the sum of all subproblem bounds is now less than z^*, go to step 10. Otherwise, perform one of the following two operations, as required by the option chosen in step 3:

a. Calculate and store both up and down subproblem bounds for each fixed first stage variable. For basic variables, the bounds are given by $\bar{z}$ minus the respective penalties. For nonbasic variables, zero and the relative costs serve in place of penalties, and bounds are calculated as for basic variables.

b. Using the optimal multipliers from step 4, form a surrogate constraint as a weighted sum of the subproblem's given constraints, and store this constraint.

If all subproblems have not been updated, return to step 4.

7. If all first stage variables have been assigned values, go to step 11. If not, and option (a) has been chosen in steps 3 and 6, go to step 9.

8. Form a linear program from the subproblem surrogate constraints determined in step 6(b), reimposing the identity of corresponding first stage variables by suppressing the first subscript in each $x_{\ell j}$ ($\ell = 1, \ldots, L$). The objective function for this surrogate LP is that of the original, unrelaxed problem (sum of subproblem objectives). Set all variables in S at their assigned values, and solve the surrogate LP. If the optimum objective value, $\hat{z}$, is less than z^*, or the problem is infeasible, go to step 10. Otherwise, determine the largest Gomory penalty, $\hat{P}_G$, among the basic first and second stage variables. If $\hat{z} - \hat{P}_G \leq z^*$, go to step 10. If fathoming still has not occurred, calculate up and down penalties for all basic first stage variables.

9. Using one of the following two methods, check for currently-unfixed first stage variables which must take on specific values if the present partial assignment is to lead to an improved integer solution:

a. For each variable, check the sum of its subproblem up bounds and the sum of its subproblem down bounds from step 6(a). If both bounds are exceeded by z^*, the current assignment is fathomed -- go to step 10. Otherwise, set all variables having one bound less than z^* to their alternate values, add the assignments to S, note that the alternate branches are fathomed, and return to step 4.

b. Using the surrogate LP penalties and relative costs from step 8, calculate up and down bounds from $\hat{z}$ for all unfixed variables. For all variables with bounds less than z^*, assign their alternate values in S, note that the other branches are fathomed, and return to step 4.

If no additional variables are fixed in this manner, return to step 3.

10. The current assignment of first stage variables has been fathomed. Referring to the ordered set S, locate the most recently-added variable whose alternate zero-one assignment has not yet been investigated, and whose associated bound exceeds z^*. Assign the alternate value to this variable and indicate (by setting its bound to $-\infty$) that both assignments have been tried. Delete from S all variables which were added subsequent to the one being reassigned, and go to step 4.

If no variable is reassigned in the above manner, terminate. The best integer solution currently in storage is optimal. If no such solution is stored ($z^* = -\infty$), the problem is infeasible.

11. Select the next subproblem, ℓ ($\ell = 1, \ldots, L$), to be solved to its integer optimum (relative to the first stage assignments in S). Set $z_\ell^* = -\infty$. If all second stage variables are naturally-integer in the last LP solution to this subproblem (step 4), go to step 14. Otherwise, go to step 16.

12. Choose as a new second stage branching variable the one having the largest, P^*, of the up and down penalties calculated in the last LP solution. Set this variable to the zero-one value opposite to P^*, add the variable to S, and store the bound on its alternate value (given by the LP objective, $\bar{z}$, minus the larger of P^* and the largest Gomory penalty, P_G^*).

13. Reoptimize the current subproblem LP while ensuring that all variables in S are nonbasic at their upper or lower bounds (as appropriate). If the subproblem is now infeasible, go to step 17. (1) If $\bar{z} \leq z_\ell^*$, go to step 17. (2) If $z_\ell^* = -\infty$, sum $\bar{z}$ with the bounds associated with all other subproblems. If this sum is less than z^* (the best known integer solution to the entire problem), go to step 17.

14. If all second stage variables have been assigned values or the current LP solution is naturally-integer, this solution is stored as the best known integer solution to the subproblem. In this case, set $z_\ell^* = \bar{z}$ and go to step 17.

15. Calculate penalties for all unfixed second stage variables in the current subproblem, and determine the largest Gomory penalty, P_G^*. Perform tests (1) and (2) of step 13, using $\bar{z} - P_G^*$ in place of $\bar{z}$.

16. Determine bounds by subtracting each non-slack basic variable's up and down penalties from $\bar{z}$, and subtracting each non-slack nonbasic variable's relative cost from $\bar{z}$. If $z_\ell^* \neq -\infty$, compare these bounds with z_ℓ^*; otherwise, compare with z^* minus the sum of bounds for all other subproblems. For each variable having a bound less than this amount, assign its alternate zero-one value, add the variable to S, and note that the other branch has been fathomed. If variables are fixed in this manner, return to step 13; otherwise, return to step 12.

17. The current assignment of second-stage variables for this subproblem has been fathomed. Locate the most recently-added variable in S whose alternate assignment has not yet been investigated, and whose bound exceeds z_ℓ^*. If $z_\ell^* = -\infty$, compare the bound with z^* minus the current bounds from all other subproblems. Set this variable to its alternate value, indicate that both assignments have been tried, delete from S all variables added subsequent to this one, and go to step 13.

If no variable is reassigned in this manner, the best integer solution in storage for this subproblem is optimum relative to the assignment of first stage variables, and z_ℓ^* is the bound to be

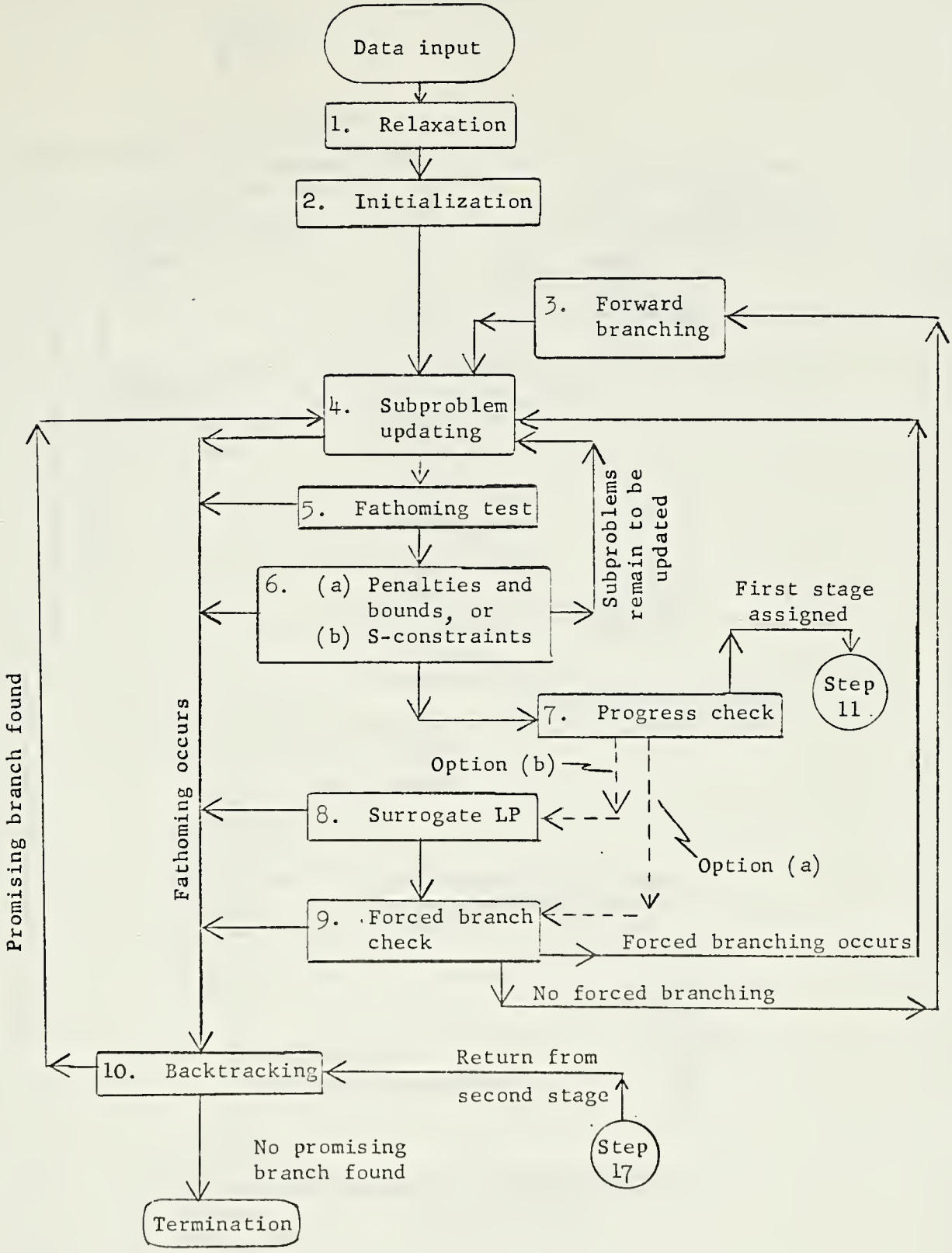


Figure 9: First Stage Flow Schematic of the Algorithm

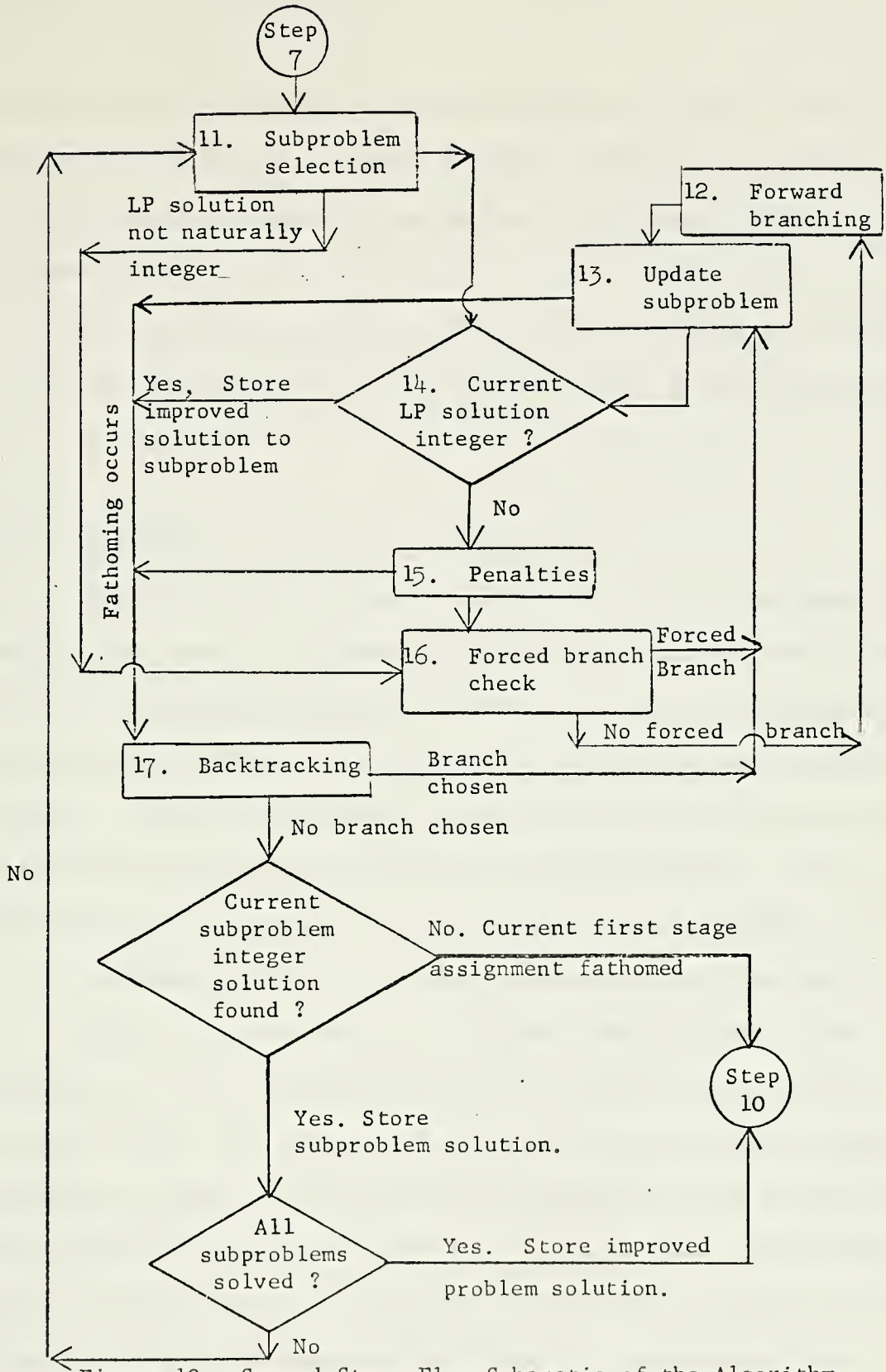


Figure 10. Second Stage Flow Schematic of the Algorithm

associated with the subproblem. If $\ell \neq L$, go to step 11. If $\ell = L$, the assignment of first and second stage variables in S gives an improved integer solution to the problem. Set $z^* = \sum_{\ell=1}^{L} z_\ell^*$, and return to step 10.

If no integer solution to the current subproblem is in storage ($z_\ell^* = -\infty$), the subproblem is integer-infeasible. Return to step 10.

5.5. Possible Modifications to the Algorithm

The literature of integer programming is rich in techniques which may be used to supplement or replace many of those used in the specific implementation described above. It would be well beyond the scope of our discussion to present a listing and discussion of these methods. Since the main body of our contribution is the technique for relaxation to independent subproblems and decomposition, rather than the penalty/surrogate LP approach used to implement this technique, we refer the reader to any of the recent surveys and texts in the field.[29]

There are several tactical considerations involved in the application of our implementation which may lead to reductions in solution times. For example, if the option to use surrogate linear programs is chosen, it may prove more economical to form a new surrogate LP less often than at each iteration of step 6. This would eliminate much of the time spent in setting up the problems and finding initial feasible solutions (simplex method, Phase I). In addition, when a single set of surrogate constraints is used for several iterations,

[29] See, for example, Geoffrion and Marsten [1972] or Garfinkel and Nemhauser [1972].

restarting techniques can be applied to update previous solutions, thereby saving time which would otherwise be required by Phase II of the simplex method. The tradeoff in this approach derives from the fact that "older" surrogate constraints may not be as strong as newly-created ones, thus resulting in reduced fathoming ability.

As mentioned in Section 3, surrogate LP's were used to supplement the fathoming test of step 6, which utilized Gomory penalties for each subproblem. Considerable savings might be realized from eliminating the calculation of subproblem penalties and relying more on the fathoming power of the surrogate linear program and its penalties. In addition, it would be quite easy to apply Balas-type logical tests to the already-available surrogate constraints when working on the assignment of first stage variables. Use of these tests when assigning second stage values would require explicit formation of the surrogate constraints from available data -- not a difficult task.

No attempt was made to identify and utilize any naturally-integer first stage variable values in the subproblems or surrogate LP. Provided all subproblems agree in their assignment of integer values to unfixed first stage variables, it is possible to fix all such variables and proceed to step 11. There is, of course, no guarantee that a naturally-integer first stage assignment in a surrogate LP will be feasible for the individual subproblems, and we conjecture that an attempt to use such assignments will be less successful than one based on subproblem solutions.

As pointed out by Schrage [1973], one of the disadvantages of a decomposition approach such as ours is that the necessity of assigning

values to all first stage variables prior to working on the second stage can lead to wasted time. This can be true if the values of several first stage variables are relatively "unimportant" in the final solution. Schrage has suggested that the flexible tree-search technique introduced by Tuan [1971] can be used to partially alleviate this problem. Given that one has identified some of these "unimportant" variables, our subproblem relaxation can provide some help in this area. If it is suspected that the current assignment of first stage variables cannot lead to an improved integer solution (regardless of what values are given to the currently-unfixed "unimportant" variables), we can suspend the restriction that all first stage variables be assigned values before any second stage variables are assigned. Branching on the second stage variables while leaving the "unimportant" first stage ones free in each subproblem may lead to faster fathoming. Unfortunately, it is not clear just how such "unimportant" variables can be recognized.

Finally, we have insisted that, after fixing all first stage variables, each successive subproblem be solved to optimality before the next one is considered. This can be excessively time-consuming if several subproblems contain some "unimportant" variables, since the various values of these variables would have to be considered before proceeding to the next subproblem. The approach might be improved by allowing partial optimization of successive subproblems in an attempt to verify infeasibility of the first stage assignment or to determine that the assignment cannot lead to an improved objective. If the attempt fails, the method can return to complete the optimization of each subproblem.

CHAPTER 6

COMPUTATIONAL RESULTS

6.1. Computer Code

Evaluation of our subproblem-decomposition algorithm was carried out on an IBM 360/67 computer, using a program written in FORTRAN IV by the author. This program was designed to facilitate experimentation with various alternatives, and therefore the results presented herein should not necessarily be considered representative of those which might be obtained from a professionally-encoded production version. In order to provide a means of comparing the relative effectiveness of our procedure, the code was designed to operate in any of the following three modes:

1. Decomposition to Independent Subproblems -- either of the two versions of our algorithm (sum-of-bounds or surrogate LP) as described in the preceding chapter.
2. Decomposition without First Stage Relaxation -- addressing the original problem formulation with a priority on first stage variables.[30]
3. No Decomposition -- straightforward solution of the problem without exploitation of structure.

Mode 2 is derived from mode 1 by simply not specifying that the values of corresponding unfixed first stage variables are allowed to differ among the subproblem LP solutions. By dropping the requirement that

[30] The decomposition method used by Schrage [1973], and discussed in Section 3.1.

all first stage variables be assigned zero-one values before any such assignments are made to second stage variables, we obtain mode 3 from mode 2.

The linear programming portion of our code was based extensively on a primal-simplex LP code with product-form inverse known as LPM-1, which was written by J.A. Tomlin of the Systems Optimization Laboratory, Department of Operations Research, Stanford University. This program was modified to use the upper bound technique for dealing with unit upper bounds on all variables, and appropriate subroutines were added to permit operation as a dual-simplex method as well. Reoptimization after forward branches was performed with primal-simplex pivots, whereas the dual-simplex method was used upon backtracking when a previously-optimal solution was available. The remainder of our code was written in functional modules according to the various tasks to be performed (branch selection, penalty calculation, testing of bounds, etc.), with a main program to coordinate the application and sequencing of the various modules. The final version of the program was compiled under IBM's FORTRAN-H compiler with the object-program optimization option employed at level two. While dimensioned to handle up to 10 subproblems of 20 rows each with 20 first stage variables and 10 second stage variables per subproblem, the object program required approximately 192 K bytes of main memory (with all instructions and data in-core).

Various options were built into the code to allow monitoring of solution paths, penalties and bounds calculated at each node, methods of branch selection and fathoming, and identification of

improved integer solutions. However, the results presented herein are of runs in which all such intermediate output (except for description of improved solutions) was suppressed. The solution times which we report are based on the total time during which the CPU was available to the core partition containing the job, excluding I/O wait time and the time required to load the object program and data in from disk storage.

6.2. Description of Testing Procedure

The primary objective of the testing program was to determine the solution-time dependence of our algorithms on the number of subproblems and constraint rows in a variety of test problems. This was accomplished by formulating a set of six "master" problems, with each such master composed of at least eight subproblems. By deleting various numbers of subproblems (and their objective function contributions) from each master, additional problems were derived and solved. Test problems obtained from any given master had the advantage of containing many of the characteristics of their source, and were therefore closely related in terms of computational complexity. This relationship made it possible to directly compare the solution times for problems of increasing size, and to attribute the results of such comparisons largely to the influence of size. Had we chosen to compare the solution times of totally unrelated problems of various sizes, it would have been necessary to generate a rather large sample in order to average-out the variations in complexity among problems of equal size.

Limitations on time and funds would necessarily have restricted our investigation to fewer types of problems if this latter alternative had been chosen.

Given a specific master problem and its derivatives, the procedure followed was to solve each member of this set using one or both variants of our decomposition algorithm. Using the solution times thus obtained as a standard, modes 2 and 3 of the preceding section were then applied to successively larger problems from the set until their time requirements grew to be excessive or all problems were solved. As will be shown in the sections dealing with each problem set, mode 1 proved to be clearly superior in all but a few of the smaller problems (with the sum-of-bounds variant outperforming the surrogate LP approach).

The sum-of-bounds variant of our algorithm experienced such a slow growth of solution times with problem size in the first six sets that a rather large test problem (30 subproblems vs. the usual 8-10) was generated and run in various sizes with mode 1. This was done in an attempt to determine if the growth rate would increase significantly as the number of subproblems was enlarged beyond the limits of previous runs. The absence of such acceleration appears to be very promising for the solution of large-scale problems by our subproblem-decomposition algorithm.

The eighth, and final, set of test problems was selected from a group derived by Schrage [1973] from a problem known as IBM-1 (Haldi [1964]). Each of these problems consisted of two subproblems having an equal number of rows and of second stage variables, and differed only in the number of first stage (or linking) variables

retained from the master problem. Even with such a small number of subproblems, our algorithm proved clearly superior to each of the other two methods tried, and showed a more consistent dependence on the number of linking variables than did Schrage's procedure.

6.3. Test Problem Number One

The master of our first test problem set was formulated from the data of an R&D project selection problem (Number 5) devised by Petersen [1967]. The original problem represented a group of 28 projects by a measure of profitability and their budgetary requirements during each of ten years. In order to obtain a problem with the structure for which our algorithm was designed, we first divided each of the original projects into two separate ones of five year's duration each. That is, the first five rows of the constraint matrix were taken to represent one group of 28 revised projects, and the last five rows represented a second group. Each objective function coefficient was divided by 10, rounded up to the nearest even integer, and then split equally among the two projects formed from its column. By alternately listing one project from each group, we obtained a collection of 56 columns of five rows each.

The first six of these columns were chosen to be the first stage matrix entries in each of the subproblems of the dual-angular structure being built. The remaining 50 columns were divided sequentially into 10 groups of 5 columns each. These ten groups

were used as the diagonal submatrices representing the second stage of our test problem. Referring to the notation of Figure 5 in Chapter 4, each of the submatrices A_i $(i = 1, \ldots, 10)$ consisted of the 5 rows and 6 columns of the first group of revised "projects", and each B_i $(i = 1, \ldots, 10)$ was a 5×5 submatrix representing one of the remaining groups of 5 projects each. Each element of the first stage cost vector, c, consisted of the sum of the identical cost rows associated with each A_i $(i = 1, \ldots, 10)$, and each d_i was the second stage cost vector associated with the corresponding B_i $(i = 1, \ldots, 10)$. The right-hand side (RHS) of each of the resulting 50 constraints was determined by taking approximately 70% of the sum of coefficients in each constraint (Petersen seems to have followed the same rule for determining the RHS of his original problems).

The cost and constraint coefficients for each of the 20 submatrices in test problem number one have been tabulated along with the aggregate first stage cost vector, c, in Table A-I in the appendix.

Five different derivative problems, consisting of the first n subproblems (n = 2, 4, 6, 8, 10) of the master, were solved using both versions of the subproblem decomposition algorithm. Attempts were also made to solve the first four of these problems with mode 2 of our code, and the first three were addressed with mode 3. The results of these efforts are shown in Table I and are plotted in Graphs I and II. "Total Nodes Explicitly Considered" refers to those nodes of the enumeration tree whose associated assignment of values to variables was actually made in the course of the solution

Table I

Results of Test Problem Number One

	Algorithm Variant	NUMBER OF SUBPROBLEMS				
		2	4	6	8	10
Total Time in Seconds	DSB	.77	2.02	11.29	29.60	27.84
	DSSC	1.43	3.21	19.38	55.39	60.38
	DFM	1.41	10.03	118.48	> 300	
	NDFM	.96	28.88	148.72		
Total Nodes Explicitly Considered	DSB	14	31	173	464	384
	DSSC	18	33	169	460	384
	DFM	20	59	366		
	NDFM	7	222	715		

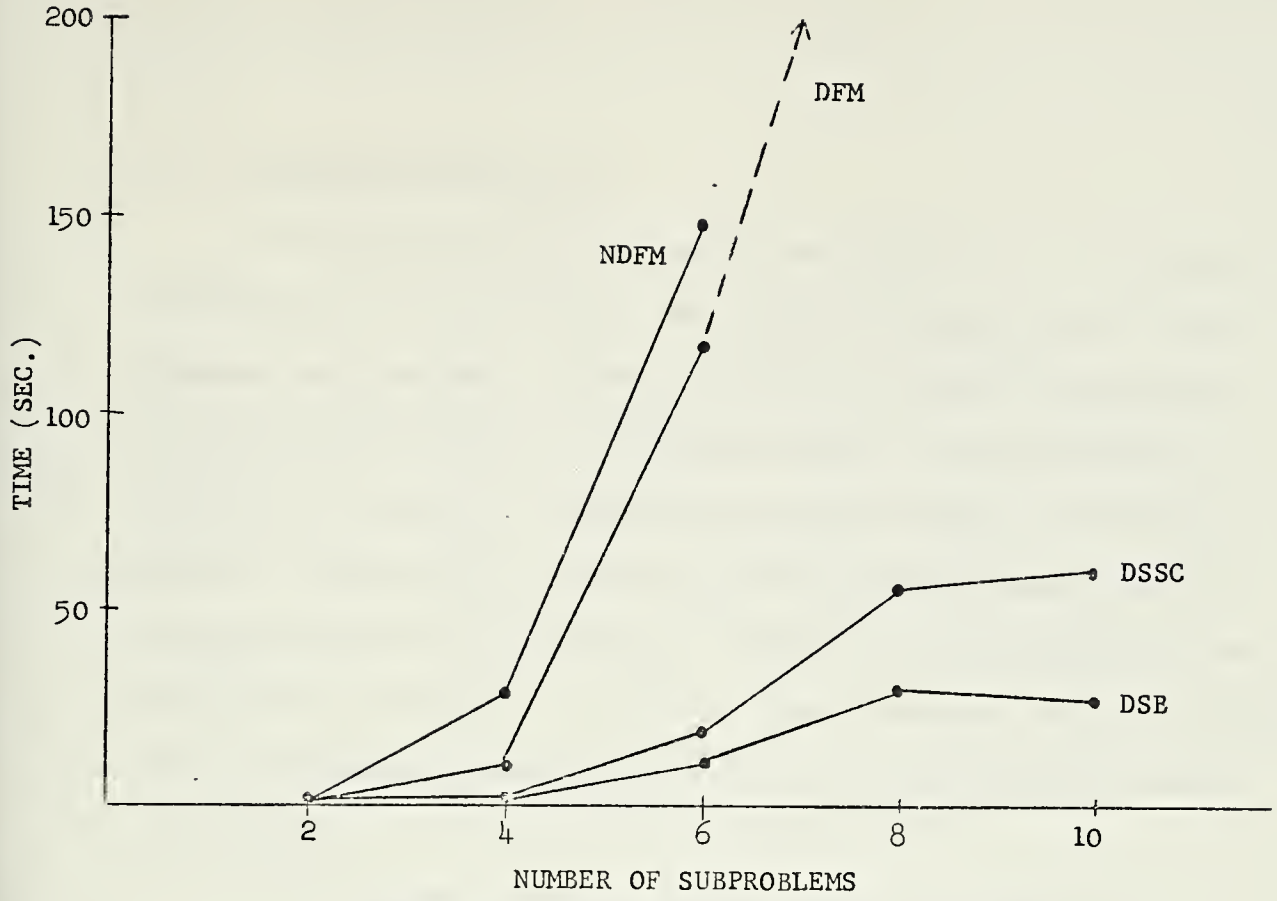
DSB = Mode 1: Sum-of-bounds variant

DSSC = Mode 1: Surrogate LP variant

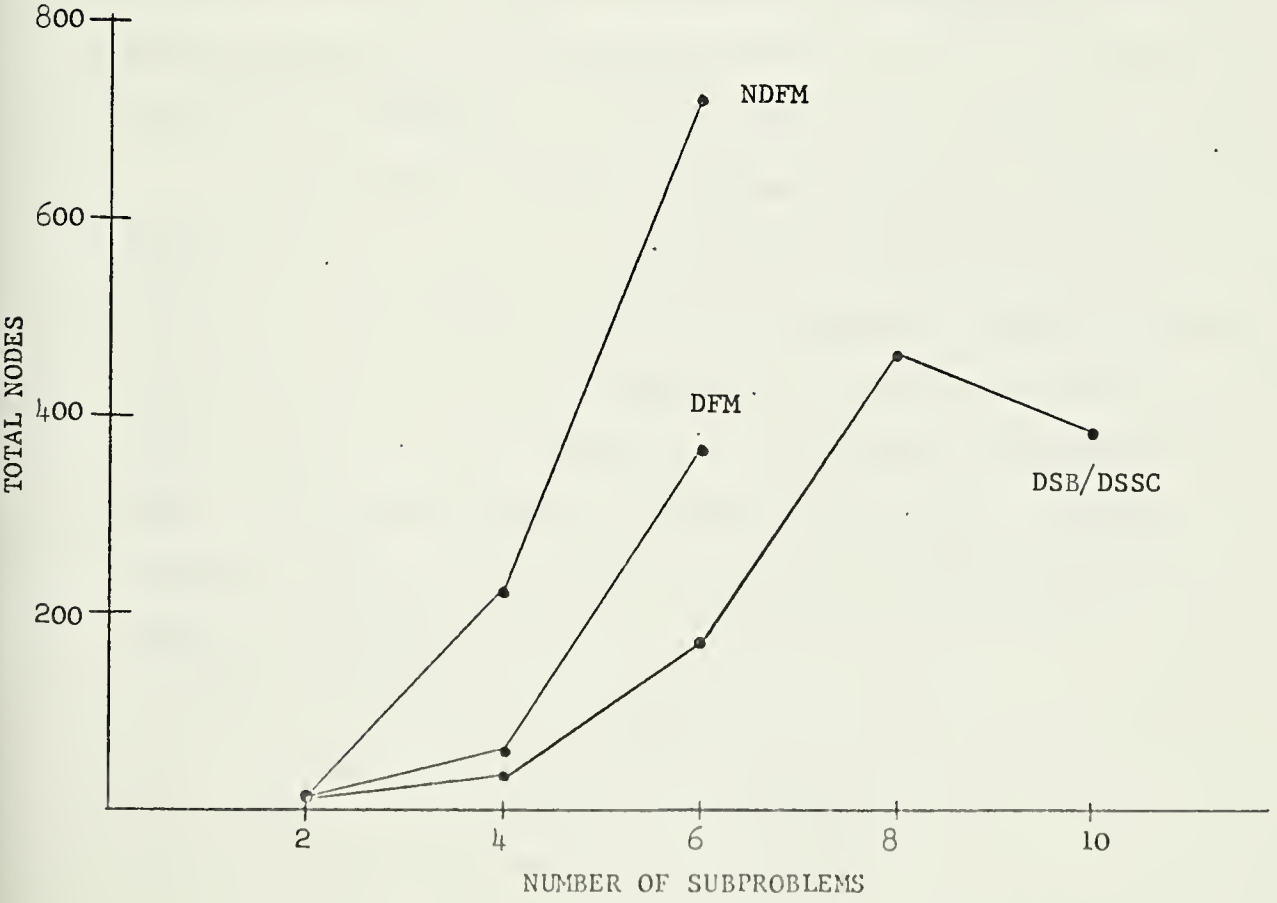
DFM = Mode 2

NDFM = Mode 3

process. Those nodes which were not analyzed explicitly because of predecessor nodes being fathomed are excluded from this count. Each of the algorithms was allowed to take advantage of the fact that all costs were integers by rounding all upper bounds obtained on various branches down to their closest integer values before testing for fathoming.



GRAPH I: Time Results, Test Problem One



GRAPH II: Node Results, Test Problem One

6.4. Test Problem Number Two

The second test problem's master was formulated from the data of Petersen's 5×50 Problem 7 in a manner similar to that of the preceding test problem. Instead of choosing the first six columns as first stage variables, we randomly selected 10 of the 50 columns, discarded the first 4 of these, and used the remaining 6 as linking variables. The other 40 columns were then divided sequentially into 8 groups of 5 and used in 8 diagonal submatrices (second stage). The original objective function coefficients were retained for all variables, and the RHS was once again determined by taking approximately 70% of each row sum.

The ten randomly-selected columns were 1, 7, 9, 14, 19, 31, 35, 38, 40, and 47. The cost and constraint coefficients of each of the 16 submatrices of test problem number two have been tabulated along with the aggregate first stage cost vector in Table A-II.

Four different derivative problems, consisting of the first n subproblems ($n = 2, 4, 6, 8$) of the master were solved by both versions of the subproblem decomposition algorithm. The first three of these were also solved with modes 2 and 3, and an attempt was made to solve the fourth using mode 3. The results are shown in Table II and plotted in Graphs III and IV. Once again, since all costs were integer, rounding was performed on all bounds before attempting fathoming.

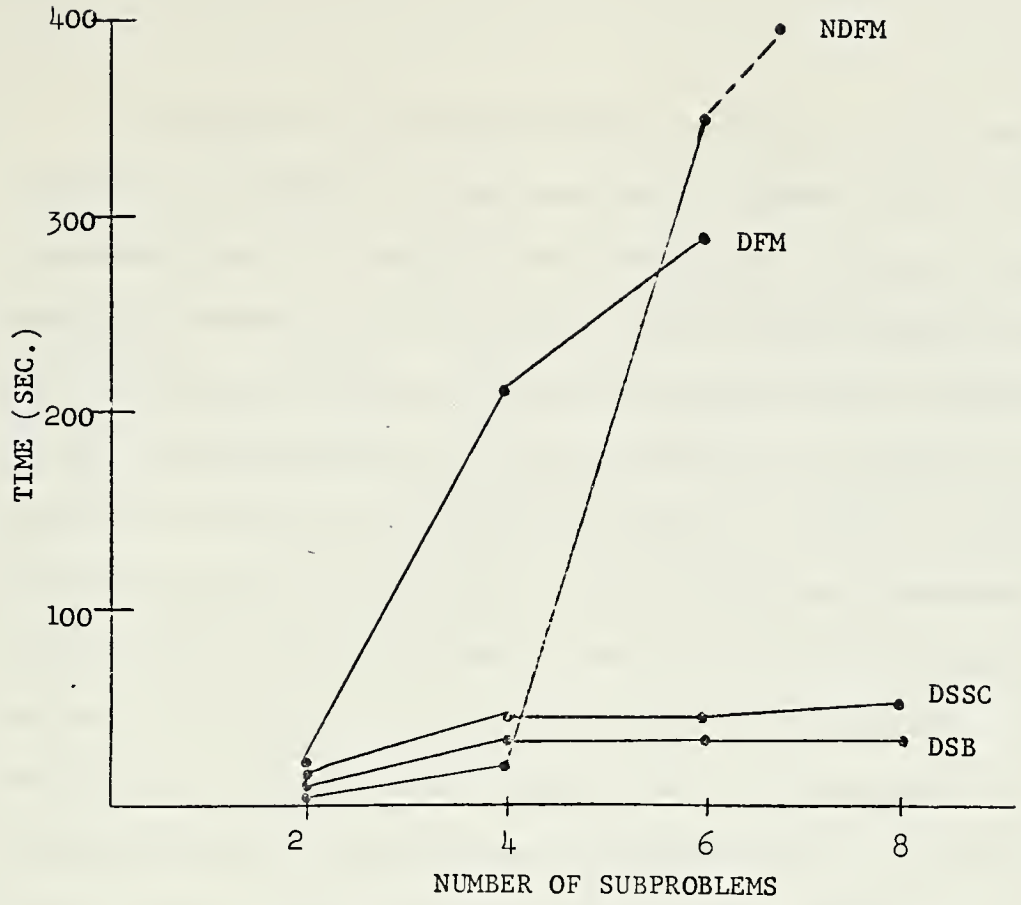
Table II

Results of Test Problem Number Two

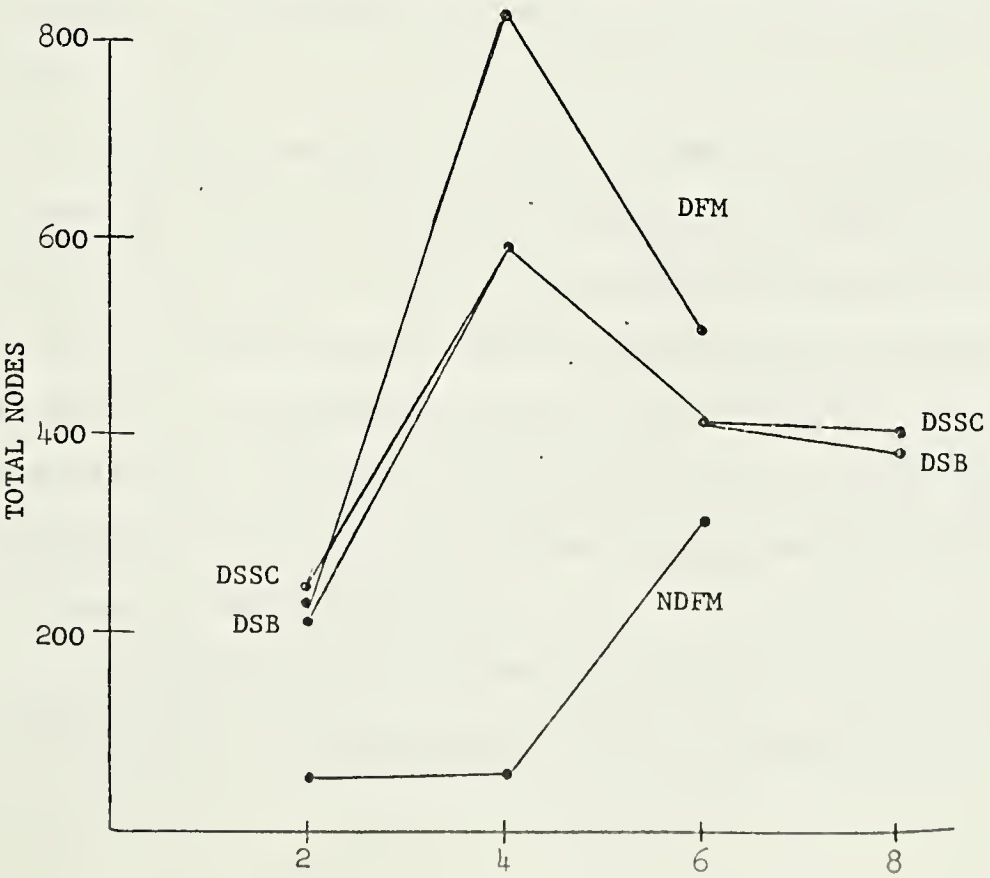
	Algorithm Variant	NUMBER OF SUBPROBLEMS			
		2	4	6	8
Total Time in Seconds	DSB	12.50	35.97	34.07	32.38
	DSSC	16.99	46.86	48.26	54.31
	DFM	24.35	212.20	290.39	
	NDFM	9.36	21.28	352.75	> 480
Total Nodes Explicitly Considered	DSB	216	592	417	382
	DSSC	245	590	422	403
	DFM	224	825	504	
	NDFM	52	57	309	

6.5. Test Problems Three and Four

Since our decomposition algorithm was initially designed to be applied to a class of multi-stage capital budgeting problems, it seemed highly desirable to use some test problems representing this class. Unfortunately, we were unable to find any dual-angular capital budgeting problems in the literature, so a matrix generating program was developed to produce "projects" with random cash flows from specified ranges. The first two such test problems were designed with 4 first stage and 4 second stage variables per subproblem, and the number of possible first stage outcomes (subproblems) was set at 10. These 10 outcomes represented 5 different general states (bad, fair, average, good, excellent) with 2 possible sets of available second stage projects for each such general state.



GRAPH III: Time Results, Test Problem Two



GRAPH IV: Node Results, Test Problem Two

The matrix was generated by first choosing for the "average" case 4 first stage project cash flow sequences as random integers uniformly distributed over the ranges shown in Table III. The first element of each such sequence was designated to be an initial "start-up" cost for its associated project, and was then added to the second element to give total first-year cost for the project. The remaining elements were then used as net cash flows for years 2, 3, and 4 (all projects had a fixed life of four years). The next step was to generate 4 second stage projects in a similar manner, except that they were designated to begin during year 2 and extend through year 5. The second set of possible second stage project availabilities for the "average" case was determined by choosing the first two such projects from the first set and generating two new ones. Therefore, there were two projects which would be available in either case at stage two.

Having accomplished the above, we were in possession of all the constraint coefficients needed to represent the "average" case (sub-matrices A_5/A_6, B_5, and B_6). The remaining four cases were derived from this "average" case by adding randomly-selected increments (or decrements) to each element of the 10 distinct projects, except for the first-year cash flows of first stage projects. These integer increments and decrements were chosen from uniform distributions over the ranges shown in Table III.

Probabilities for occurrence of the five major states were chosen to be 0.1, 0.2, 0.4, 0.2, and 0.1, respectively, from "bad"

Table III

Test Problems Three through Six: Parameters

Time of Flow	Initial Range	Increment Case and Range			
		Bad	Fair	Good	Excellent
Start-up	[-40, -25]				
Year 1	[-25, 0]	[-9, -3]	[-7, 2]	[-2, 7]	[3, 9]
Year 2	[-25, 0]	[-9, -3]	[-7, 2]	[-2, 7]	[3, 9]
Year 3	[30, 50]	[-9, -3]	[-7, 2]	[-2, 7]	[3, 9]
Final Year	[20, 40]	[-9, -3]	[-7, 2]	[-2, 7]	[3, 9]

to "excellent". Within each of these states, probabilities of 0.4 and 0.6 were assigned to the two possible second stage project availabilities, resulting in probabilities of (0.04, 0.06, 0.08, 0.12, 0.16, 0.24, 0.08, 0.12, 0.04, 0.06) for the ten subproblems. Objective function "weights" were determined for the subproblems by dropping the decimals in these probabilities and dividing by their greatest common denominator (2). The subproblem objective function coefficients were then determined by summing each project's cash flows and multiplying by the appropriate objective function "weight".

The final step was to determine a RHS for each constraint in the "average" case by adding 35% of the sum of its positive coefficients to 65% of the sum of its negative ones. The RHS's thus obtained for the two project-availabilities were then averaged and rounded, and this final set of values was used as the budgetary limits (RHS) for every subproblem. All constraints were "greater-than-or-equal-to" inequalities.

The cost and constraint coefficients of test problems three and four are listed in Tables A-III and A-IV.

Each of these two problems was run in five sizes formed by successively removing the subproblems corresponding to the bad, excellent, fair, and good outcomes, in that order. All 10 problems were solved with modes 1 (sum-of-bounds variant), 2, and 3 of our algorithm. The results of these efforts are presented in Tables IV and V and in Graphs V - VIII. Rounding of all bounds down to nearest integers was employed throughout.

Table IV

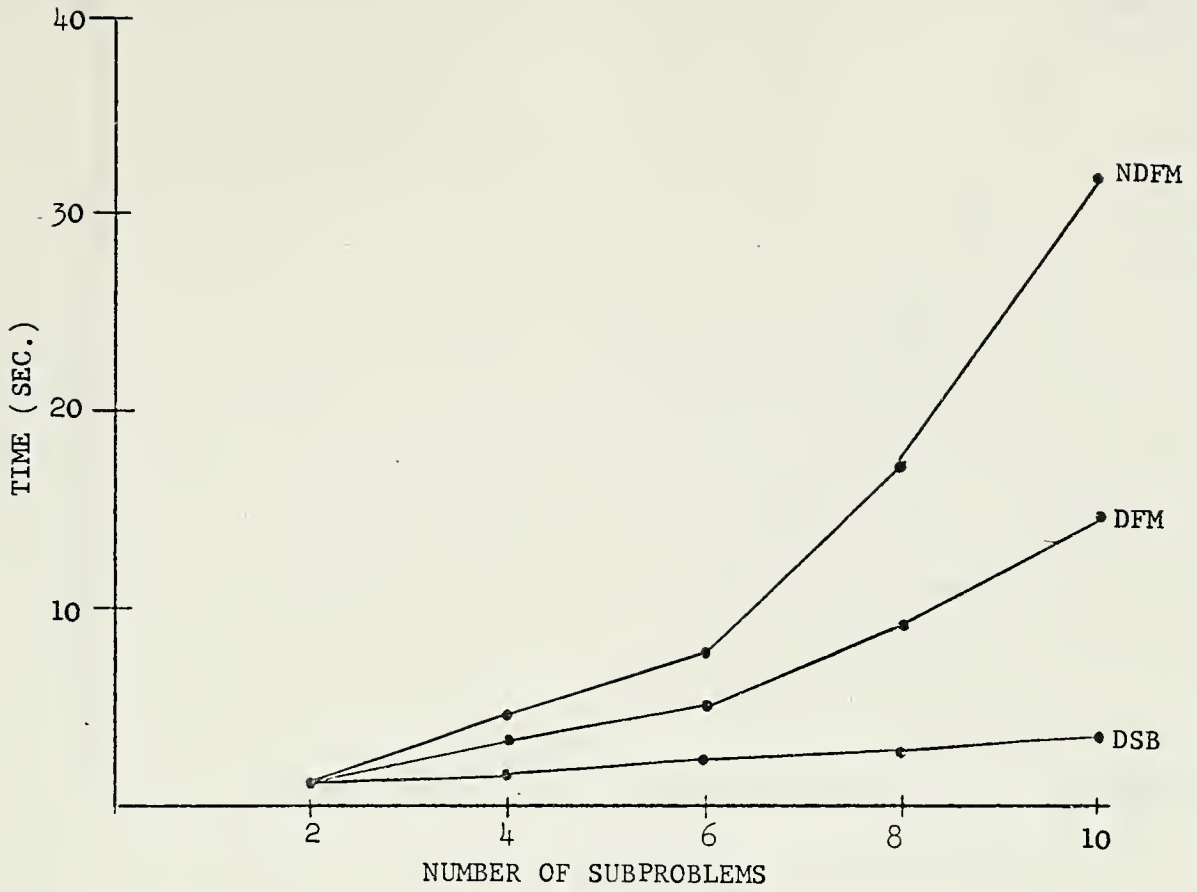
Results of Test Problem Number Three

	Algorithm Variant	NUMBER OF SUBPROBLEMS				
		2	4	6	8	10
Total Time in Seconds	DSB	1.03	1.64	2.38	2.95	3.59
	DFM	1.17	3.27	5.09	9.09	14.81
	NDFM	1.32	4.71	7.76	17.18	31.85
Total Nodes Explicitly Considered	DSB	21	25	28	35	37
	DFM	22	28	28	36	40
	NDFM	29	55	55	94	135

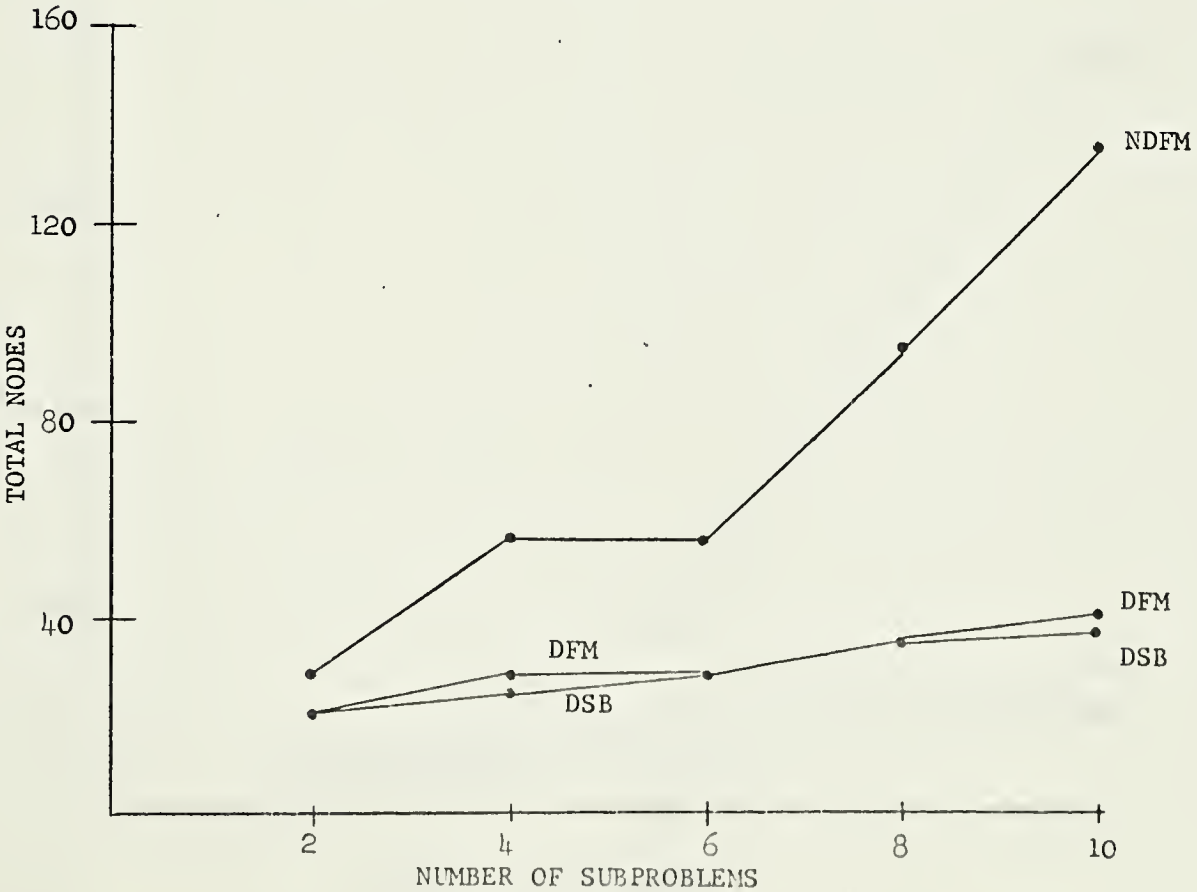
Table V

Results of Test Problem Number Four

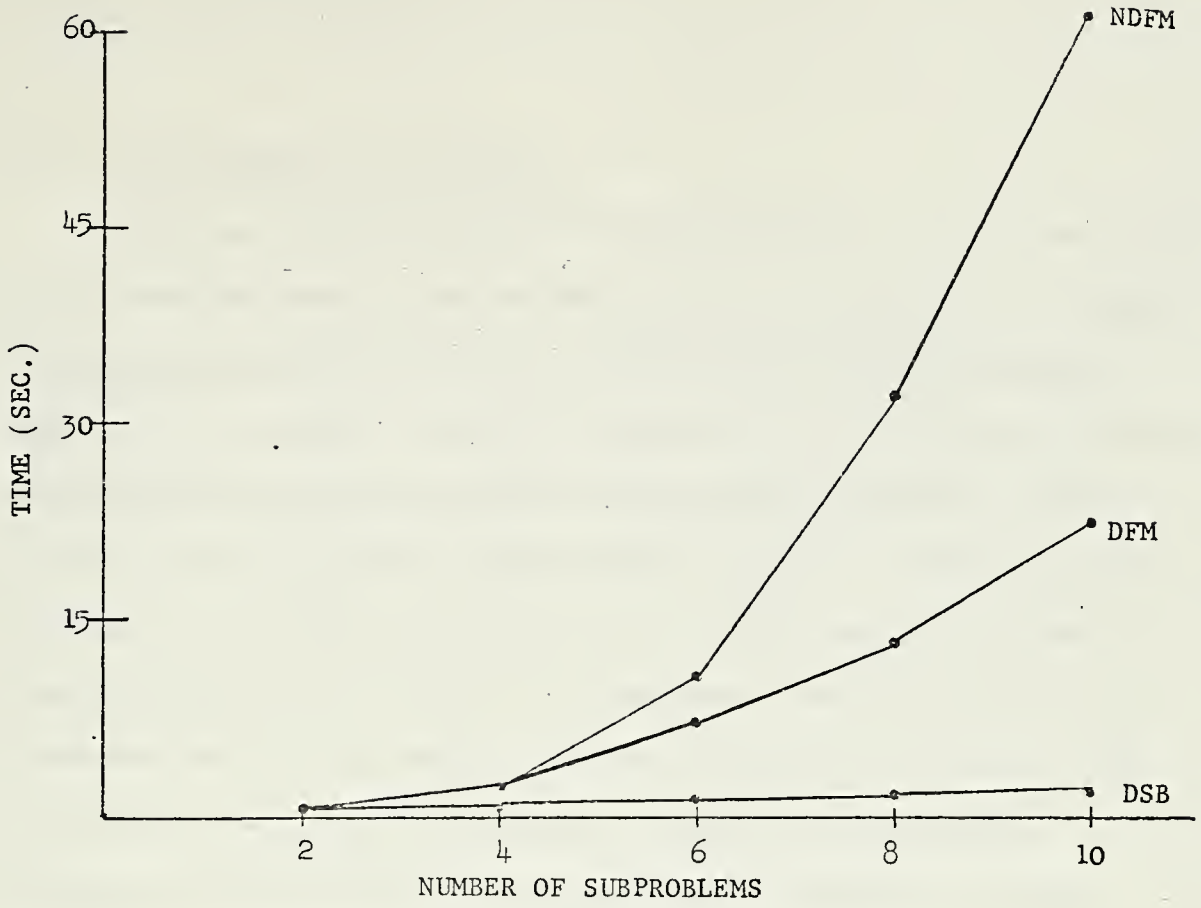
	Algorithm Variant	NUMBER OF SUBPROBLEMS				
		2	4	6	8	10
Total Time in Seconds	DSB	.34	.80	1.19	1.61	2.09
	DFM	.72	2.64	7.57	13.20	22.25
	NDFM	.57	2.76	10.60	32.68	61.95
Total Nodes Explicitly Considered	DSB	9	16	22	29	38
	DFM	12	21	31	39	49
	NDFM	8	24	69	163	269



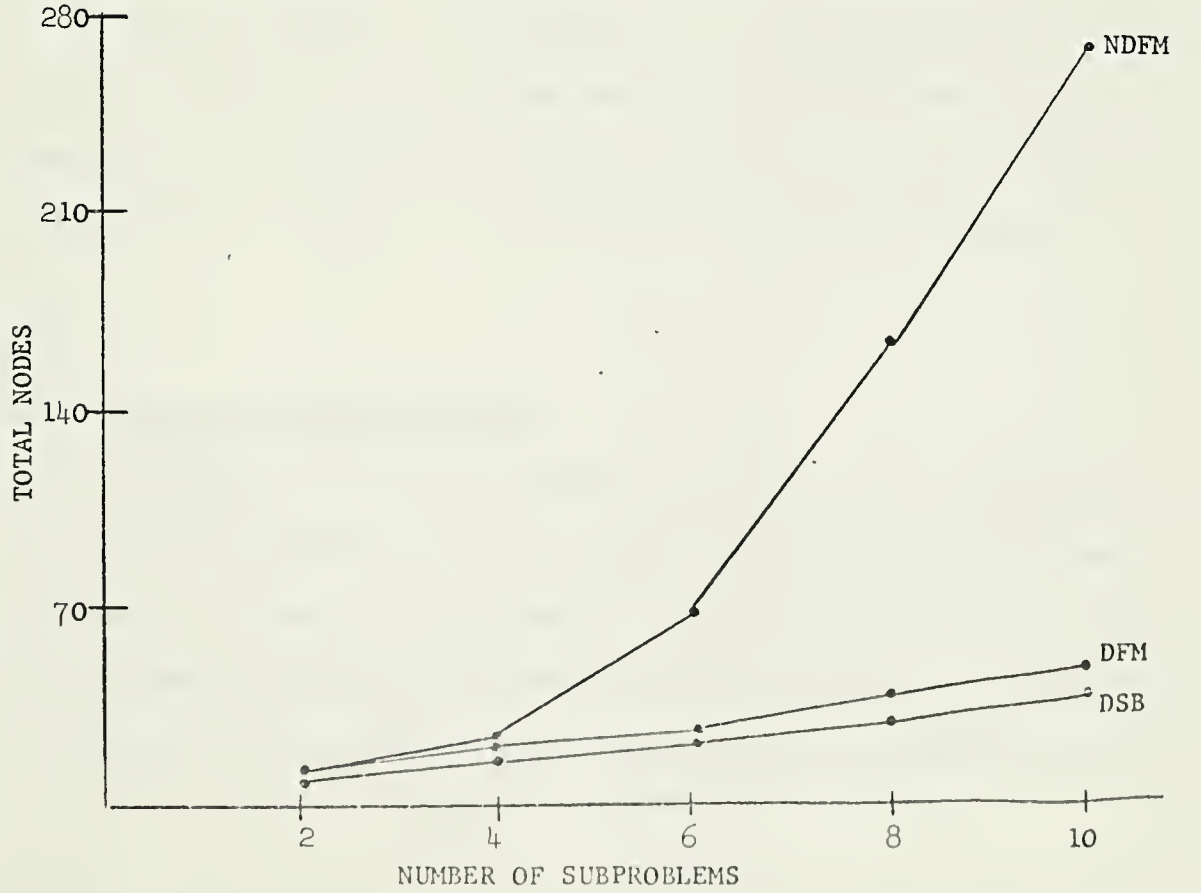
GRAPH V: Time Results, Test Problem Three



GRAPH VI: Node Results, Test Problem Three



GRAPH VII: Time Results, Test Problem Four



GRAPH VIII: Node Results, Test Problem Four

6.6. Test Problems Five and Six

The very encouraging results obtained by our subproblem decomposition algorithm (DSB) in problems three and four, especially the somewhat less-than-proportional increase of solution times with the number of subproblems attempted, led to the formulation of two more such problems. The main difference in these additional problems was that they were designed with 5 projects in each stage instead of 4. The same matrix generator and parameters (Table III) were used along with the same probabilities for each of the 10 subproblems. The only difference in procedure was that the RHS was calculated using 45% and 75% (vs. 35% and 65%) of the positive and negative constraint coefficients, respectively. The objective and constraint coefficients of these problems are presented in Tables A-V and A-VI.

These additional test problems were run using the same algorithm variants applied to the preceding two, and the results are given in Tables VI and VII and Graphs IX - XII. Integer objective coefficients again allowed all bounds to be rounded down to nearest integers.

6.7. Test Problem Number Seven

Experience with the test problems discussed above (especially problems 3-6) indicated a marked tendency for the solution times and node counts to increase approximately linearly with problem size when the subproblem-decomposition algorithm (DSB) was applied. In retrospect, this result is not particularly surprising since the algorithm largely

Table VI

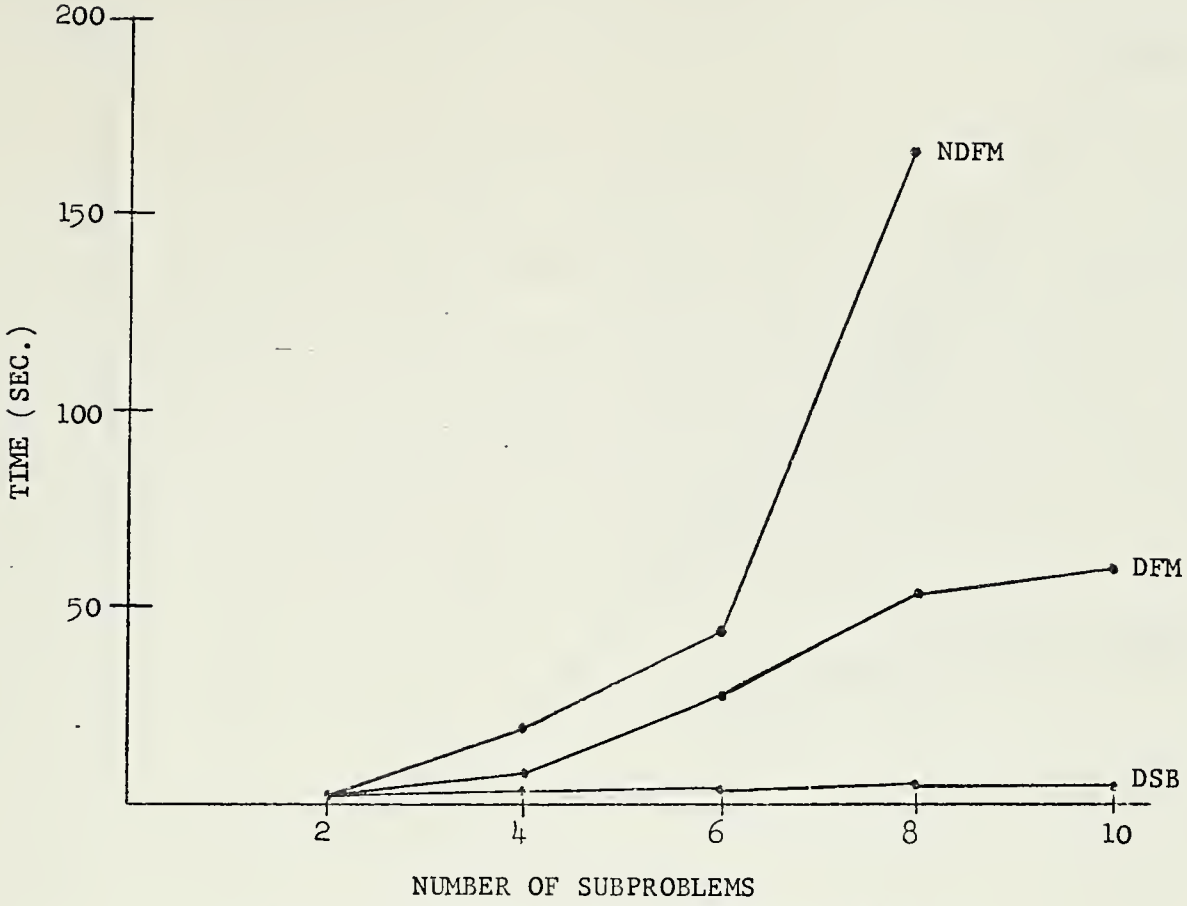
Results of Test Problem Number Five

	Algorithm Variant	NUMBER OF SUBPROBLEMS				
		2	4	6	8	10
Total Time in Seconds	DSB	1.38	2.39	3.34	5.06	5.00
	DFM	1.84	7.07	28.06	54.76	60.65
	NDFM	1.79	19.29	44.08	166.30	--
Total Nodes Explicitly Considered	DSB	29	31	34	47	45
	DFM	25	51	139	183	128
	NDFM	26	194	288	818	--

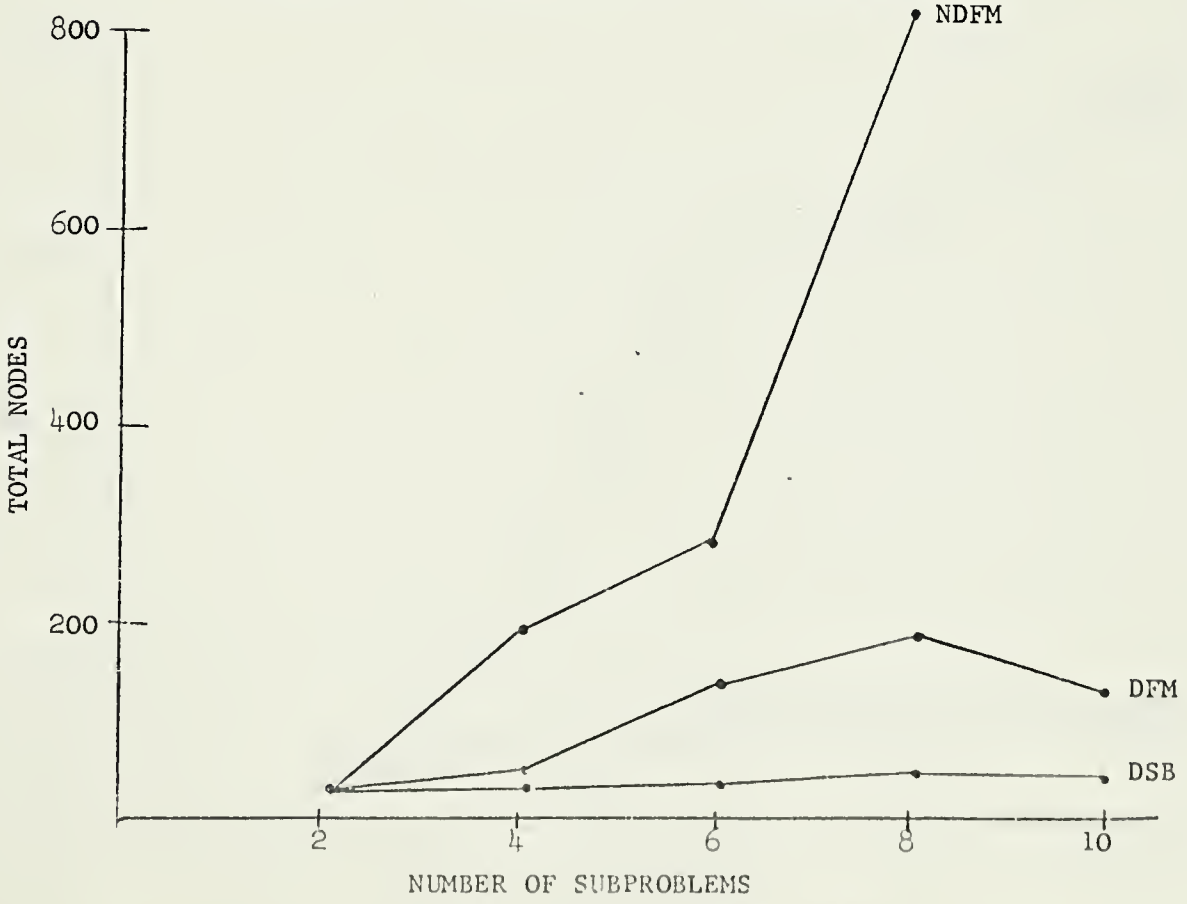
Table VII

Results of Test Problem Number Six

	Algorithm Variant	NUMBER OF SUBPROBLEMS				
		2	4	6	8	10
Total Time in Seconds	DSB	.62	1.31	2.10	2.32	2.71
	DFM	1.10	3.47	10.22	15.20	18.35
	NDFM	.87	6.68	30.69	45.04	77.00
Total Nodes Explicitly Considered	DSB	16	26	34	34	37
	DFM	19	30	39	39	43
	NDFM	13	73	224	225	273



GRAPH IX: Time Results, Test Problem Five



GRAPH X: Node Results, Test Problem Five

GRAPH XI: Time Results, Test Problem Six

GRAPH XII: Node Results, Test Problem Six

ignores the interrelationships among subproblems and treats each additional one as a separate integer program to be solved in parallel with the rest. In effect, doubling the total size of a problem requires not the periodic solution of a linear program which is twice as large, but the periodic solution of twice as many linear programs of the same size. The same relationship holds true with respect to calculation of penalties and performance of the various bookkeeping and overhead operations. The fact that the last four problems gave rise to more consistent results than the first two may be attributed to the formers' similarity among subproblems, as was noted in Section 2 of this chapter. Such similarities would be expected in "real-world" multi-stage capital budgeting problems, which our matrix generator was designed to simulate.

In order to obtain additional information on the relationship between problem size and computational effort with our algorithm, the matrix generator was used to produce a large problem (30 subproblems) with characteristics identical to those of test problems 3 and 4. Each subproblem consisted of 6 rows (including the objective), 4 first-stage variables, and 4 second stage variables. The problem was run in 8 different sizes using the subproblem-decomposition method (DSB), and the results are given in Table VIII and Graphs XIII and XIV.

The outcome of this additional test series obviously reinforces our contention that the subproblem-decomposition algorithm will generally experience a linear growth of solution time with problem size when applied to dual-angular structures of the type herein considered, especially if there exists a similarity or consistency

Table VIII

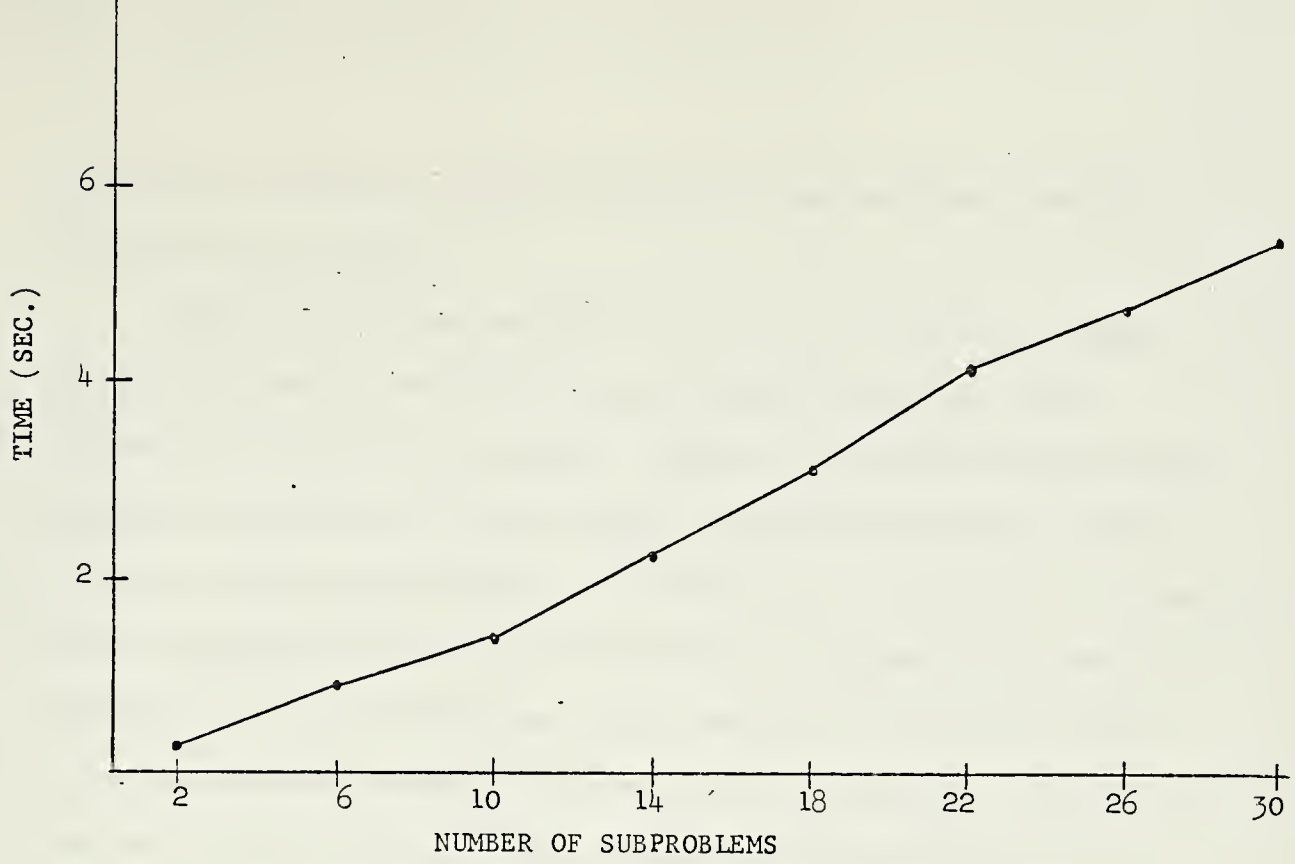
Results of Test Problem Number Seven

	NUMBER OF SUBPROBLEMS							
	2	6	10	14	18	22	26	30
Total Time in Seconds	.34	.91	1.40	2.24	3.10	4.18	4.79	5.51
Total Nodes Explicitly Considered	8	12	15	24	33	46	50	54

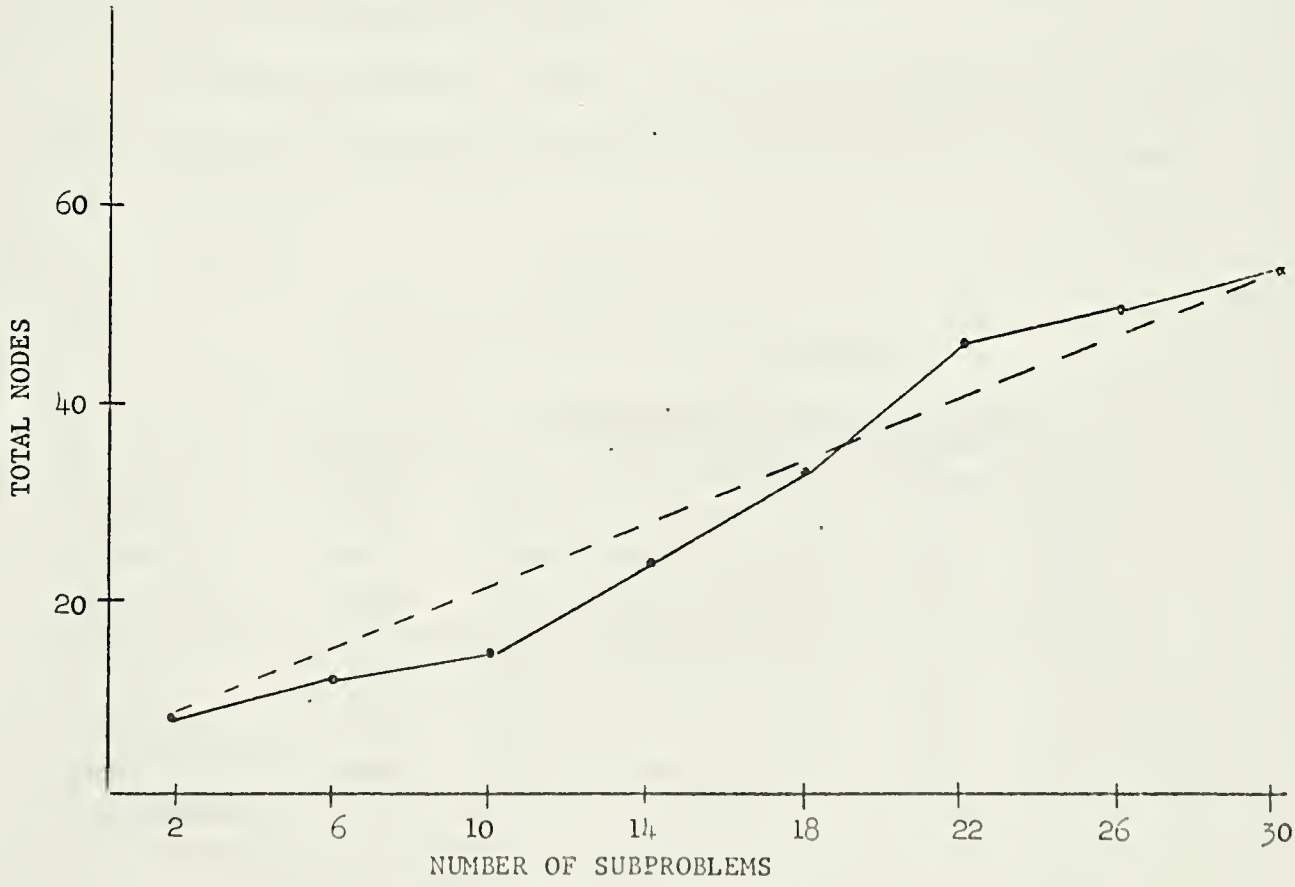
among subproblems. Additional testing of this hypothesis on a wider range of problem types would of course be desirable, but indications are that the method will provide a viable technique for the solution of large-scale dual-angular integer programs.

6.8. Schrage's Test Problem

When applied to the two-subproblem versions of our first six test problems, the subproblem-decomposition algorithm obtained results quite similar (and sometimes inferior) to those obtained by modes 2 and 3 of the code. Despite the fact that the algorithm's relative efficiency could be questioned at this lower level, it was decided to apply the method to a selection of 5 of the 8 test problems devised by Schrage [1973]. The main purpose of this investigation was to gain insight into the behavior of the three algorithm modes as we increased the number of linking variables (first stage) while holding



GRAPH XIII: Time Results, Test Problem Seven



GRAPH XIV: Node Results, Test Problem Seven

the number of subproblems (two) and second stage variables (fourteen per subproblem) constant.

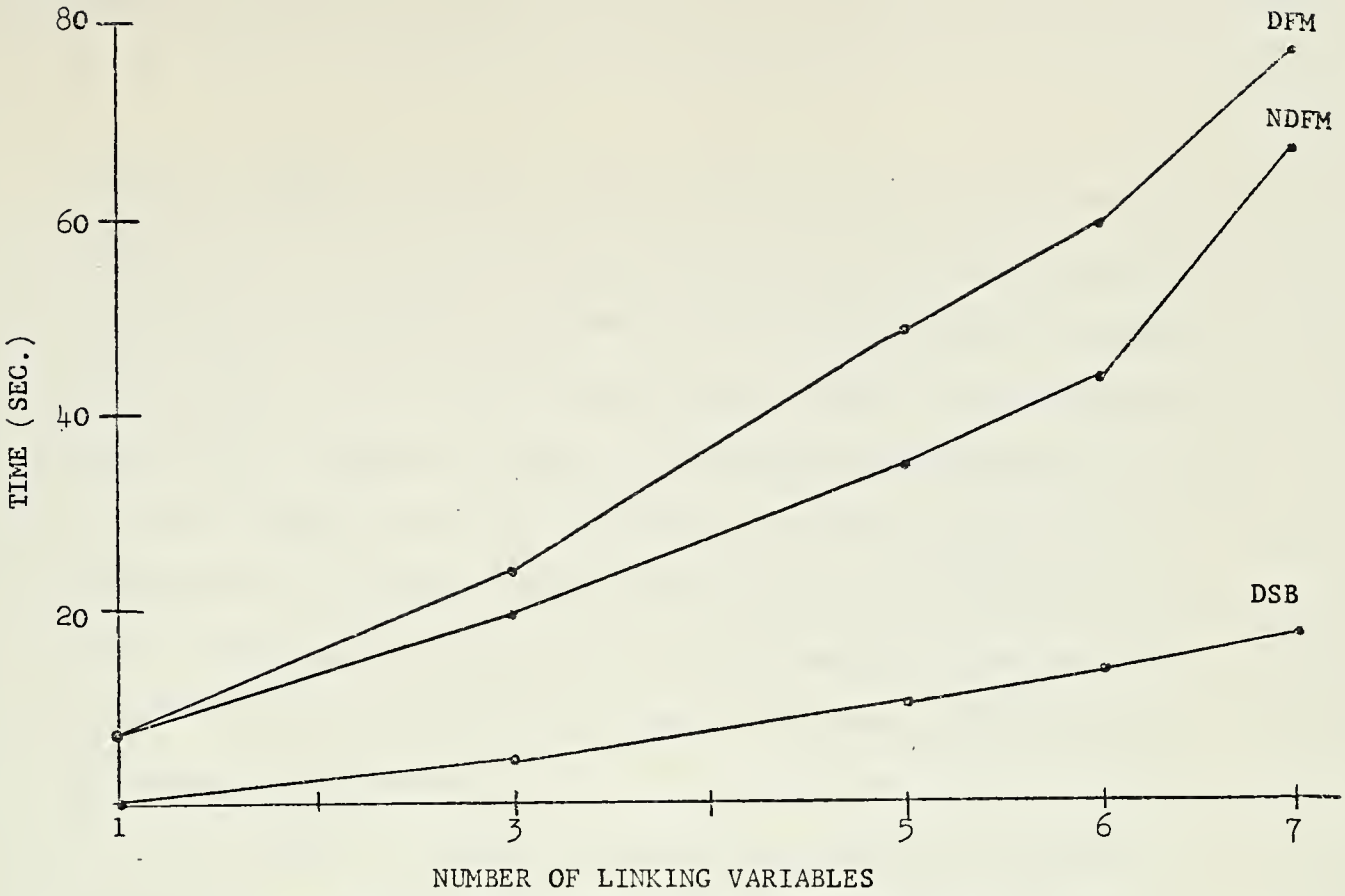
Since there existed no natural allocation of each first stage variable's objective coefficient among the two subproblems, equal division was arbitrarily specified. Rather than doubling the resulting subproblem objectives to obtain integer costs, we chose not to enable the bound-rounding feature used in previous runs. In addition to other tactical differences between our algorithm and Schrage's, this lack of rounding in our runs renders any direct comparison of solution times or node counts with Schrage's reported results meaningless. We therefore base our conclusions solely on the relative performances of our algorithm's three modes, and draw no conclusions concerning the efficiency of our method vs. Schrage's.

The results reported in Table IX and shown in Graphs XV and XVI indicate that the subproblem-decomposition algorithm again outperformed

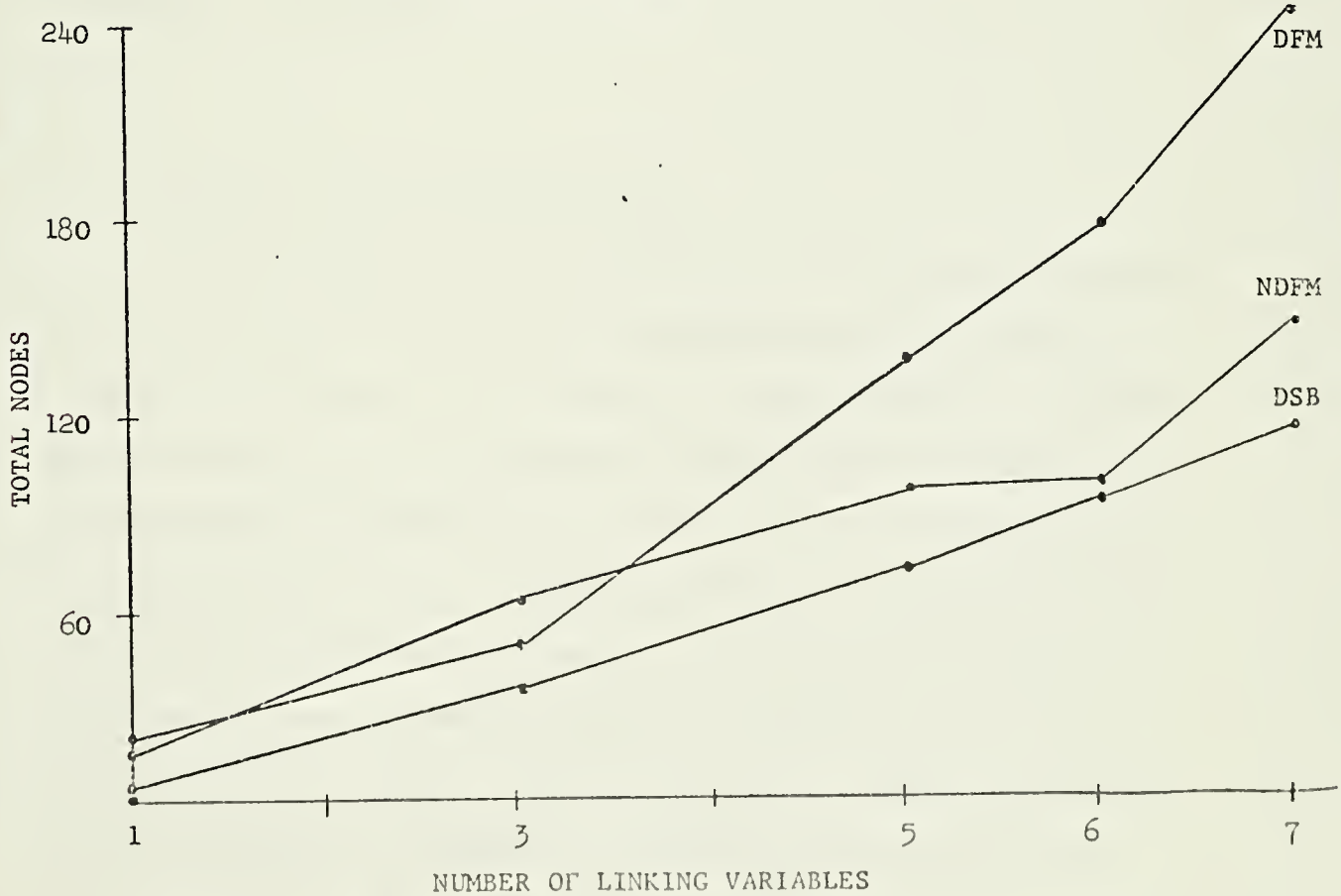
Table IX

Results of Schrage's Test Problem

	Algorithm Variant	NUMBER OF LINKING VARIABLES				
		1	3	5	6	7
Total Time in Seconds	DSB	1.09	4.51	9.97	13.41	16.90
	DFM	7.80	23.43	47.90	58.60	76.87
	NDFM	6.97	19.63	34.02	43.04	66.63
Total Nodes Explicitly Considered	DSB	7	34	70	92	114
	DFM	21	48	135	176	243
	NDFM	17	61	95	96	145



GRAPH XV: Time Results, Schrage's Test Problem Without Rounding



GRAPH XVI: Node Results, Schrage's Test Problem Without Rounding

both other methods, but with a fairly constant ratio of solution times. The increase of solution times with the number of linking variables is approximately linear with each method, particularly that of subproblem-decomposition. It is also interesting to note that the method without decomposition (NDFM) had a higher node count than the straight decomposition method (DFM) in only one instance, and that its solution times were consistently better. This might be construed as indicating that Schrage's problem(which has only two subproblems) is not particularly amenable to such decomposition, and might therefore explain his somewhat mixed results with it.

The results of this test series give no reason to conclude that decomposition loses its effectiveness as the number of linking variables is increased.

6.9. Summary of Computational Results

Results obtained from the previously described testing procedure strongly indicate that the subproblem-decomposition algorithm enjoys significant advantages over the other two methods tested. The solution times experienced with this method are quite commonly 50 - 80% below those obtained from mode 2 (DFM), which also relies on decomposition via priority specification of linking variables but performs all its operations on the full problem matrix. In the majority of cases with our first six test problems, both of these algorithms have provided much better results than a straightforward method taking no advantage of structure (NDFM). Comparison of the node counts obtained with

DSB/DSSC vs. DFM indicate that, rather than causing decreases in fathoming ability, the separation of subproblems actually may result in an improved capability to recognize unpromising first and second stage branches. It is believed that the obvious weakening of objective bounds which occurs when subproblems are uncoupled is often compensated by the advantage of using each subproblem's largest Gomory penalty rather than a single such penalty taken from the full linear programming tableau. The advantage of having strong bounds for each subproblem seems to be particularly pronounced after a problem has decomposed naturally and integer optima are sought for the independent subproblems. Reference to steps 13 and 16 of our algorithm (Chapter 5, Section 5.4) will clarify the reasons for this claim.

The surrogate-LP version of our algorithm consistently required more time and considered more nodes than its companion sum-of-bounds approach. A separate timer was placed on that portion of the code which set up, optimized, and calculated penalties for the surrogate LP. The time required for these efforts accounted for approximately 50-85% of the additional solution times for DSSC vs. DSB. The remaining time can be attributed to consideration of additional nodes. As was discussed in Sections 5.3 and 5.5, there exist several possible methods for reducing the time required by our surrogate LP option and enhancing its fathoming capabilities; however, it is doubtful that such procedures can give this option a significant advantage over the sum-of-bounds approach.

Our algorithm's approximately linear growth of solution times and node counts with problem size (as determined by the number of

subproblems considered) make it an extremely promising technique for the solution of large-scale dual-angular integer linear programs.

CHAPTER 7

CONCLUSIONS AND AREAS FOR FUTURE RESEARCH

7.1. Capital Budgeting Formulation

As presented in Chapter 3, our extension of Hillier's basic two-stage model to allow consideration of uncertain future availability of projects adds another dimension of realism to a promising theoretical technique for the solution of multi-period capital budgeting problems. In the past, many mathematical programming approaches to capital budgeting have relied on an assumption that all present and future investment opportunities (within the time frame being considered) could be identified and incorporated in the analysis. The applicability of such techniques to "real world" situations has been limited by that assumption because, in most situations, very few possible future events can be regarded as being "certain" to occur. In an environment of uncertainty, one can impose constraints on future cash positions to provide a source of funds for possible investment in desirable opportunities which might arise. This type of approach is rather artificial because it requires that the amounts to be kept available be determined outside of the solution process for the model, thereby reducing or eliminating explicit analytical use of whatever partial information on future events may be available.

At the price of expanding Hillier's model and increasing its computational complexity, we are able to incorporate some of this partial

information directly in the analysis. This is accomplished by relaxing the aforementioned assumption to require predictions of which projects may become available and estimations of their associated probabilities. In practice, of course, consideration would be given only to those projects, or combinations of projects deemed to have the most influence on a model's behavior.

Requirements for predictions of possible investment opportunities and estimations of probabilities demand additional management input to an analysis and more thorough understanding by analysts of the uncertain environment which they are attempting to idealize. Philosophically speaking, this is a very desirable secondary benefit of utilizing the proposed approach. From a more practical point of view, solution of a large model with explicit consideration of future uncertainties should produce much more useful results than would the independent processing of a large number of small problems based on rigid assumptions about the future. Given that the latter approach is taken, one is left with the formidable task of resolving a single "best" course of immediate action from a set of indications which may differ considerably because of their underlying assumptions. This problem is even more difficult in a capital budgeting context, wherein many or all decisions are of the "accept-reject" variety, and therefore not amenable to compromise.

Having argued the case for a more refined and computationally-complex analysis, it is necessary to address the problem of solving the resulting formulation. We believe that the subproblem-decomposition method presented in Chapters 4 through 6 will prove to be a very

efficient technique for the solution of that class of integer programs which includes our formulation of multi-period capital budgeting with uncertain future project availabilities.

7.2. Subproblem-Decomposition Method

The computational experience reported in Chapter 6 strongly indicates that our technique of "uncoupling" column-linked subproblems and then using an enumeration procedure to gradually retrieve the original problem structure is quite efficient for the solution of dual-angular integer programs. Our specific implementation of the method within the framework of a penalty-type algorithm resulted in substantial improvements over both of the other previously-developed techniques which were tested. These improvements were noted not only in total solution times, but also in the numbers of nodes explicitly considered by the various methods. Of equal, or perhaps greater, importance is the approximately linear growth of solution times with the number of subproblems that was experienced with the subproblem-decomposition algorithm. Assuming that this observed behavior is indeed characteristic of the relaxation procedure, the computational tractability of large-scale dual-angular integer linear programs will be determined mainly by that of their component subproblems.

7.3. Areas for Future Research

The field of capital budgeting contains many unresolved problems for both financial experts and operations researchers. Development of procedures for determining the complex interrelationships among investment proposals and incorporating these into valid mathematical models, especially when non-linearities and discontinuities are involved, merits additional study. The same is true of methods for incorporating attitudes toward risk into mathematical idealizations of the decision process; present practices generally fall into two categories -- those which are acknowledged to be somewhat artificial but relatively easy to implement (e.g., chance-constrained programming), and those which have a sound theoretical foundation but cannot be readily obtained or employed (e.g., utility functions). Also, the difficulty of determining a firm's exact "cost of capital" for use in analysis may be at least partially avoided by developments in the area of post-optimality sensitivity analysis for integer programming.

We have already given considerable attention in Chapters 4 and 5 to possible alternatives and modifications in our subproblem-decomposition method. As mentioned previously, the method can be applied within the frameworks of most popular implicit enumeration algorithms, and it would be desirable to obtain indications of its effectiveness in a number of different procedures. In particular, the surrogate constraint algorithm developed by Geoffrion [1969] appears to be quite efficient and would be a prime candidate for synthesis with our method. Additional development of the use of subproblem

surrogate constraints and surrogate linear programs might well prove to be a fertile area for research. It should also be noted that, although we have restricted our discussions to problems involving binary variables, the method can be extended in a direct way to deal with general integer variables.

The majority of test problems used in this study have had equal, or nearly-equal, numbers of first and second stage variables per subproblem. Although such a situation would be common with capital budgeting problems, other types of models (such as those in which stages do not correspond to successive time periods) may not possess such a structure. For this reason, it would be interesting to consider a broader range of test problems of two and more stages. In particular, the behavior of our method in two-stage problems when the number of second stage variables is gradually reduced to zero, if it remains successful, would indicate that problems with many rows might be solved more efficiently if first relaxed to a number of smaller "subproblems" and then reunited by the enumeration process. The question of how to assign subproblem objective coefficients in such a situation (and others in which no natural allocation exists) also deserves serious attention.

In Chapter 3, we discussed four cases in which continuous decision variables appearing prior to the completion of a problem's decomposition could be handled with little difficulty. We also pointed out that continuous variables which appear only in the final level of decomposition create no problems for the decomposition

approach. Obviously, the development of techniques for employing decomposition in problems with general continuous variables not covered by these cases would be of tremendous value for solving dual-angular mixed integer programs.

Finally, for other types of structured integer programs, it is hoped that the success of our technique will kindle greater interest in pre-solution relaxations which are progressively retightened as the enumeration process evolves. We suggest the block-angular structure with linking rows as one to which such a technique might be applied.

APPENDIX

Table A-I

Test Problem Number One

A_i		COLUMNS 1	2	3	4	5	6
	c_i	5	5	11	11	5	5
	1	8	5	24	11	13	7
R	2	8	0	44	0	13	1
O	3	3	3	6	4	4	5
W	4	5	3	9	6	6	9
	5	5	3	11	8	7	9

$(i = 1, \ldots, 10)$

$c = (50, 50, 110, 110, 50, 50)$

B_1		COLUMN 1	2	3	4	5	RHS
	d_1	20	20	15	15	20	
	1	80	55	70	40	80	275
R	2	100	10	100	4	90	260
O	3	20	20	20	14	30	90
W	4	40	30	30	29	40	145
	5	50	35	40	29	40	165

B_2		COLUMN 1	2	3	4	5	RHS
	d_2	20	10	10	6	6	
	1	40	45	21	15	9	195
R	2	10	75	0	25	6	200
O	3	20	8	6	3	12	65
W	4	20	16	12	5	12	100
	5	20	19	16	7	15	120

B_2		COLUMN 1	2	3	4	5	RHS
	d_3	8	8	29	29	20	
R	1	28	18	90	29	130	310
O	2	28	0	120	6	130	315
W	3	12	10	14	18	40	95
	4	18	10	24	30	60	155
	5	18	10	29	30	70	175

B_4		COLUMN 1	2	3	4	4	RHS
	d_4	20	7	7	5	5	
R	1	70	32	21	20	17	215
O	2	32	32	3	40	0	190
W	3	42	6	9	3	12	80
	4	42	16	18	11	18	130
	5	42	21	20	17	18	150

B_5		COLUMN					
		1	2	3	4	5	RHS
	d_5	65	65	33	33	16	
	1	120	35	40	25	30	280
R	2	160	70	40	10	60	355
O	3	20	100	5	20	0	130
W	4	30	110	25	20	10	190
	5	30	120	25	20	15	210

B_6		COLUMN					
		1	2	3	4	5	RHS
	d_6	16	24	24	4	4	
	1	20	20	25	6	5	160
R	2	0	55	0	10	0	160
O	3	5	5	6	3	4	48
W	4	15	13	18	5	7	95
	5	20	25	22	5	7	120

B_7		COLUMN					
		1	2	3	4	5	RHS
	d_7	3	3	130	130	155	
	1	3	2	180	110	220	465
R	2	6	0	240	0	290	490
O	3	0	1	20	20	30	80
W	4	1	2	80	40	60	180
	5	1	3	100	50	70	220

B_8		COLUMN					
		1	2	3	4	5	RHS
	d_8	155	55	55	48	48	
	1	70	50	55	30	20	260
R	2	30	80	10	90	0	265
O	3	50	40	30	10	5	90
W	4	60	50	50	20	25	200
	5	60	55	55	20	25	215

B_9		COLUMN					
		1	2	3	4	5	RHS
	d_9	23	23	15	15	11	
	1	50	50	12	20	5	200
R	2	70	10	27	10	17	210
O	3	0	20	5	20	0	65
W	4	30	25	10	25	5	120
	5	50	30	15	25	15	160

B_{10}		COLUMN					
		1	2	3	4	5	RHS
	d_{10}	11	10	10	26	26	
	1	15	8	6	18	20	150
R	2	5	8	0	28	10	310
O	3	10	0	10	10	20	65
W	4	15	3	10	20	28	110
	5	15	6	10	20	28	120

Table A-II

Test Problem Number Two

A_i	COLUMN 1	2	3	4	5	6
c_i	82	49	71	116	738	620
1	1	0	4	10	40	28
2	2	4	3	0	28	45
3	3	5	8	0	6	14
4	0	2	1	20	14	20
5	6	0	3	5	0	22

$(i = 1,\ldots,8)$

$c = (656, 392, 586, 928, 5904, 4960)$

B_1	COLUMN 1	2	3	4	5	RHS
d_1	1125	300	620	2100	431	
1	91	10	30	160	20	275
R 2	92	41	16	150	23	280
O 3	39	32	71	80	26	205
W 4	71	30	60	200	18	300
5	52	30	42	170	9	235

B_2	COLUMN 1	2	3	4	5	RHS
d_2	328	122	322	196	41	
1	12	18	9	25	1	100
R 2	18	0	12	8	2	85
O 3	40	12	30	15	0	100
W 4	30	8	31	6	3	90
5	20	3	21	4	1	55

B_3		COLUMN					
		1	2	3	4	5	RHS
	d_3	425	4260	416	115	22	
	1	10	280	10	8	1	275
R	2	0	200	20	6	1	315
O	3	23	100	0	20	0	130
W	4	18	60	21	4	2	110
	5	14	310	8	4	1	260

B_4		COLUMN					
		1	2	3	4	5	RHS
	d_4	631	132	420	86	42	
	1	49	8	21	6	1	115
R	2	70	9	22	4	1	130
O	3	40	6	8	0	6	75
W	4	32	15	31	2	2	95
	5	18	15	38	10	4	85

B_5		COLUMN					
		1	2	3	4	5	RHS
	d_5	103	215	81	91	26	
	1	5	10	8	2	1	75
R	2	5	10	6	4	0	75
O	3	4	22	4	6	1	55
W	4	7	8	2	8	0	55
	5	8	6	0	0	3	35

B_6		COLUMN					
		1	2	3	4	5	RHS
	d_6	420	316	72	49	108	
	1	10	42	6	8	0	100
R	2	12	8	4	0	10	80
O	3	14	8	2	0	20	60
W	4	8	6	7	0	0	50
	5	10	6	1	0	3	35

B_7		COLUMN					
		1	2	3	4	5	RHS
	d_7	90	1811	430	3060	215	
	1	1	86	11	120	8	215
R	2	6	93	9	30	22	170
O	3	0	12	6	80	13	110
W	4	8	20	2	40	6	90
	5	4	30	12	16	18	80

B_8		COLUMN					
		1	2	3	4	5	RHS
	d_8	58	296	418	47	81	
	1	3	32	13	2	4	95
R	2	0	36	13	2	2	95
O	3	6	22	0	1	2	55
W	4	1	14	12	0	1	55
	5	3	16	30	4	0	60

Table A-III

Test Problem Number Three

c = (2250, 1725, 1025, 1945)

RHS = (-115, -116, 44, 83, 33)

ROW	COLUMN											
	A_1/A_2				B_1				B_2			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(1)	64	36	18	34	44	-22	-62	- 6				
OBJ(2)	96	54	27	51					66	-33	-45	87
1	-42	-33	-45	-43	0	0	0	0	0	0	0	0
2	6	- 5	0	4	-46	-52	-64	-65	-46	-52	-51	-39
3	41	35	30	41	3	-14	-14	3	3	-14	-11	5
4	27	21	24	15	42	41	24	30	42	41	28	42
5	0	0	0	0	23	14	23	29	33	14	19	21

ROW	COLUMN											
	A_3/A_4				B_3				B_4			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(3)	120	116	64	116	140	0	-32	52				
OBJ(4)	180	174	96	174					210	0	36	288
1	- 42	- 33	- 45	- 43	0	0	0	0	0	0	0	0
2	6	- 3	1	12	- 39	- 49	- 63	- 64	- 39	- 49	- 41	- 32
3	43	38	34	44	1	- 15	- 6	4	1	- 15	- 12	7
4	23	27	26	16	42	46	33	39	42	46	34	45
5	0	0	0	0	31	18	28	34	31	18	25	28

ROW	COLUMN											
	A_5/A_6				B_5				B_6			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(5)	368	272	168	320	400	120	0	160				
OBJ(6)	552	408	252	480					600	180	180	672
1	- 42	- 33	- 45	- 43	0	0	0	0	0	0	0	0
2	10	4	5	11	- 38	- 44	- 59	- 57	- 38	- 44	- 42	- 32
3	48	38	34	50	7	- 10	- 6	7	7	- 10	- 7	13
4	30	25	27	22	49	48	33	38	49	48	37	47
5	0	0	0	0	32	21	32	32	32	21	27	28

ROW	COLUMN											
	A_7/A_8				B_7				B_8			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(7)	220	168	80	180	260	148	48	148				
OBJ(8)	330	252	120	270					390	222	132	396
1	- 42	- 33	- 45	- 43	0	0	0	0	0	0	0	0
2	15	6	5	10	- 33	- 37	- 59	- 52	- 33	- 37	- 39	- 31
3	52	37	32	55	14	- 4	- 1	7	14	- 4	- 9	18
4	30	32	28	23	52	54	36	45	52	54	41	45
5	0	0	0	0	32	24	36	37	32	24	29	34

ROW	COLUMN											
	A_9/A_{10}				B_9				B_{10}			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(9)	128	98	80	128	140	74	62	92				
OBJ(10)	192	147	120	192					210	111	108	258
1	- 42	- 33	- 45	- 43	0	0	0	0	0	0	0	0
2	16	7	10	19	- 29	- 40	- 50	- 50	- 29	- 40	- 33	- 29
3	51	45	39	57	11	- 6	- 1	13	11	- 6	- 4	22
4	39	30	36	31	52	53	42	43	52	53	41	56
5	0	0	0	0	36	30	40	40	36	30	32	37

Table A-IV

Test Problem Number Four

C = (2730, 1740, 1110, 1055)

RHS = (-121, -94, 68, 94, 35)

ROW	COLUMN											
	A_1/A_2				B_1				B_2			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(1)	66	22	10	- 2	60	16	36	- 2				
OBJ(2)	99	33	15	- 3					90	24	- 60	21
1	-34	- 54	- 53	- 45	0	0	0	0	0	0	0	0
2	13	11	- 2	- 7	- 45	- 50	- 47	- 37	- 45	- 50	- 58	- 50
3	38	40	37	27	16	1	- 1	- 17	16	1	- 3	3
4	16	14	23	24	40	39	30	28	40	39	24	40
5	0	0	0	0	19	18	36	25	19	18	17	14

ROW	COLUMN											
	A_3/A_4				B_3				B_4			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(3)	216	88	68	32	144	64	148	36				
OBJ(4)	324	132	102	48					216	96	- 42	156
1	- 34	- 54	- 53	- 45	0	0	0	0	0	0	0	0
2	21	16	4	- 6	- 45	- 44	- 46	- 30	- 45	- 44	- 56	- 47
3	43	45	37	34	16	- 3	8	- 12	16	- 3	3	8
4	24	15	29	25	44	41	40	30	44	41	29	45
5	0	0	0	0	21	22	35	21	21	22	17	20

ROW	COLUMN											
	A_5/A_6				B_5				B_6			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(5)	440	296	168	176	416	216	360	144				
OBJ(6)	660	444	252	264					624	324	60	396
1	- 34	- 54	- 53	- 45	0	0	0	0	0	0	0	0
2	19	20	6	- 1	- 41	- 43	- 40	- 32	- 41	- 43	- 50	- 44
3	46	49	41	36	20	4	8	- 9	20	4	1	11
4	24	22	27	32	49	45	38	31	49	45	30	46
5	0	0	0	0	24	21	39	28	24	21	24	20

ROW	COLUMN											
	A_7/A_8				B_7				B_8			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(7)	224	180	120	136	236	152	212	156				
OBJ(8)	336	270	180	204					354	228	66	240
1	- 34	- 54	- 53	- 45	0	0	0	0	0	0	0	0
2	21	22	9	- 2	- 35	- 38	- 41	- 29	- 35	- 38	- 51	- 42
3	46	53	42	43	25	5	10	- 4	25	5	- 1	15
4	23	24	32	38	47	49	38	37	47	49	32	47
5	0	0	0	0	22	22	46	35	22	22	31	20

ROW	COLUMN											
	A_9/A_{10}				B_9				B_{10}			
	1	2	3	4	1	2	3	4	1	2	3	4
OBJ(9)	146	110	78	80	144	104	134	76				
OBJ(10)	219	165	117	120					216	156	93	195
1	- 34	- 54	- 53	- 45	0	0	0	0	0	0	0	0
2	25	27	13	4	- 38	- 40	- 36	- 23	- 38	- 40	- 42	- 36
3	49	54	48	44	23	10	11	- 6	23	10	10	20
4	33	28	31	37	58	53	47	34	58	53	36	53
5	0	0	0	0	29	29	45	33	29	29	27	[illegible]

Table A-V

Test Problem Number Five

c = (1970, 1285, 1270, 2040, 1615)

RHS = (-196, -148, 105, 170, 60)

ROW	COLUMN														
	A_1/A_2					B_1					B_2				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(1)	46	20	12	50	30	- 10	34	46	34	18					
OBJ(2)	69	30	18	75	45						-15	51	-15	9	-39
1	49	-57	-64	-37	-54	0	0	0	0	0	0	0	0	0	0
2	14	- 6	11	- 2	11	-62	-52	-36	-42	-52	-62	-52	-54	-49	-70
3	36	43	31	42	28	- 7	7	0	1	- 5	- 7	7	5	- 2	4
4	22	30	28	22	30	42	40	37	24	46	42	40	28	40	39
5	0	0	0	0	0	22	22	22	34	20	22	22	16	14	14

ROW	COLUMN														
	A_3/A_4					B_3					B_4				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(3)	128	88	96	108	104	36	80	204	116	92					
OBJ(4)	192	132	144	162	156						54	120	102	162	-30
1	-49	-57	-64	-37	-54	0	0	0	0	0	0	0	0	0	0
2	13	- 3	19	0	19	-59	-50	-29	-40	-50	-59	-50	-44	-44	-71
3	42	46	37	43	27	- 2	5	6	5	2	- 2	5	13	4	3
4	26	36	32	21	34	49	42	47	32	47	49	42	27	44	46
5	0	0	0	0	0	21	23	27	32	24	21	23	21	23	17

ROW	COLUMN														
	A_5/A_6					B_5					B_6				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(5)	336	208	184	336	240	136	360	400	336	256					
OBJ(6)	504	312	276	504	360						204	540	252	324	144
1	-49	-57	-64	-37	-54	0	0	0	0	0	0	0	0	0	0
2	18	3	19	6	17	-56	-43	-29	-39	-48	-56	-43	-46	-45	-65
3	44	46	35	48	32	- 2	10	5	10	3	- 2	10	11	7	10
4	29	34	33	25	35	49	49	46	33	49	49	49	33	43	46
5	0	0	0	0	0	26	29	28	38	28	26	29	23	22	21

ROW	COLUMN														
	A_7/A_8					B_7					B_8				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(7)	156	116	132	208	168	76	212	240	172	180					
OBJ(8)	234	174	198	312	252						114	318	186	282	114
1	-49	-57	-64	-37	-54	0	0	0	0	0	0	0	0	0	0
2	17	2	24	12	18	-56	-42	-29	-40	-44	-56	-42	-40	-42	-59
3	44	51	33	48	37	3	17	8	11	7	3	17	10	14	10
4	27	33	40	29	41	48	49	47	31	52	48	49	36	49	49
5	0	0	0	0	0	24	29	34	41	30	24	29	25	26	19

ROW	COLUMN														
	A_9/A_{10}					B_9					B_{10}				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(9)	122	82	84	114	104	82	138	132	132	122					
OBJ(10)	183	123	126	171	156						123	207	135	147	102
1	-49	-57	-64	-37	-54	0	0	0	0	0	0	0	0	0	0
2	27	12	27	10	26	-48	-36	-26	-30	-44	-48	-36	-39	-38	-61
3	50	49	41	53	40	5	16	11	17	11	5	16	17	11	13
4	33	37	38	31	40	53	53	50	37	58	53	53	37	48	55
5	0	0	0	0	0	31	36	31	42	36	31	36	30	28	27

Table A-VI

Test Problem Number Six

C = (1275, 2725, 2130, 300, 900)

RHS = (-163, -153, 97, 151, 61)

ROW	COLUMN														
	A_1/A_2					B_1					B_2				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(1)	20	74	54	-38	-14	0	8	38	48	44					
OBJ(2)	30	111	81	-57	-21						0	12	12	-39	21
1	-42	-44	-34	-51	-46	0	0	0	0	0	0	0	0	0	0
2	6	13	11	- 3	-12	-52	-59	-49	-42	-46	-52	-59	-42	-52	-57
3	26	39	22	23	38	- 4	1	- 2	10	8	- 4	1	2	-10	12
4	20	29	28	12	13	42	34	43	42	33	42	34	28	31	30
5	0	0	0	0	0	14	28	27	14	27	14	28	16	18	22

ROW	COLUMN														
	A_3/A_4					B_3					B_4				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(3)	68	164	120	8	56	64	48	112	120	124					
OBJ(4)	102	246	180	12	84						96	72	132	- 6	126
1	-42	-44	-34	-51	-46	0	0	0	0	0	0	0	0	0	0
2	14	16	11	3	-10	-49	-54	-51	-42	-49	-49	-54	-38	-56	-55
3	24	37	24	29	46	- 3	3	7	14	17	- 3	3	11	- 2	16
4	21	32	29	21	24	47	33	43	38	38	47	33	36	35	31
5	0	0	0	0	0	21	30	29	20	25	21	30	13	22	29

ROW	COLUMN														
	A_5/A_6					B_5					B_6				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(5)	208	440	368	32	136	184	240	304	352	376					
OBJ(6)	312	660	552	48	204						276	360	348	36	384
1	-42	-44	-34	-51	-46	0	0	0	0	0	0	0	0	0	0
2	15	19	18	4	- 5	-45	-55	-45	-38	-42	-45	-55	-39	-49	-50
3	30	42	31	31	46	- 1	9	6	14	17	- 1	9	11	- 4	20
4	23	38	31	20	22	46	39	47	45	40	47	39	37	34	33
5	0	0	0	0	0	23	37	30	23	32	23	37	20	22	29

ROW	COLUMN														
	A_7/A_8					B_7					B_8				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(7)	124	260	184	68	108	112	136	184	236	232					
OBJ(8)	186	390	276	102	162						168	204	234	90	300
1	-42	-44	-34	-51	-46	0	0	0	0	0	0	0	0	0	0
2	20	24	16	10	0	-46	-53	-40	-36	-44	-46	-53	-40	-43	-48
3	32	44	30	38	52	- 2	10	8	17	21	- 2	10	15	- 3	24
4	21	41	34	20	21	53	40	45	48	47	53	40	41	35	40
5	0	0	0	0	0	23	37	33	30	34	23	37	23	26	34

ROW	COLUMN														
	A_9/A_{10}					B_9					B_{10}				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
OBJ(9)	90	152	126	50	74	86	100	116	144	124					
OBJ(10)	135	228	189	75	11						129	150	135	75	156
1	-42	-44	-34	-51	-46	0	0	0	0	0	0	0	0	0	0
2	22	26	24	11	- 2	-42	-52	-38	-31	-39	-42	-52	-33	-45	-41
3	34	50	38	37	54	5	17	12	17	21	5	17	14	3	25
4	31	44	35	28	31	51	44	50	54	45	51	44	41	40	5[illegible]
5	0	0	0	0	0	29	41	34	32	35	29	41	23	27	[illegible]

REFERENCES

Armstrong, R.D. and P. Sinha, "Improved Penalty Calculations for a Mixed Integer Branch-and-Bound Algorithm," School of Engineering, Department of Industrial Engineering and Operations Research and School of Business Administration, Department of General Business and Finance, Technical Report No. 8, University of Massachussets, Amherst, June 1973.

Balas, E., "An Additive Algorithm for Solving Linear Programs with Zero-One Variables," Operations Research, Vol. 13, No. 4 (July-August, 1965), pp. 517-546.

Balas, E., "Discrete Programming by the Filter Method," Operations Research, Vol. 15, No. 5 (September-October, 1967), pp. 915-957.

Baumol, W. J. and R.E. Quandt, "Investment and Discount Rates Under Capital Rationing -- A Programming Approach," Economic Journal, Vol. 75, No. 298 (June, 1965), pp. 317-329.

Benichou, M., J.M. Gauthier, P. Girodet, G. Hentgès, G. Ribière, and O. Vincent, "Experiments in Mixed Integer Linear Programming," Mathematical Programming, Vol. 1, No. 1 (October, 1971), pp. 76-94.

Bernhard, R.H., "Some Problems in the Use of a Discount Rate for Constrained Capital Budgeting," Journal of Finance and Quantitative Analysis, Vol. 2, No. 2 (June, 1969), pp. 111-158.

Byrne, R.F., A. Charnes, W.W. Cooper, and K.O. Kortanek, "A Chance-Constrained Approach to Capital Budgeting with Portfolio-Type Payback Constraints and Horizon Posture Controls," Chapter 3 in R.F. Byrne, et al., (ed.), Studies in Budgeting, North-Holland, Amsterdam, 1971a.

Byrne, R.F., A. Charnes, W.W. Cooper, and K.O. Kortanek, "C^2 and LPU^2 Combinations for Treating Different Risks and Uncertainties in Capital Budgets," Chapter 4 in R.F. Byrne, et al.,(ed.), Studies in Budgeting, North-Holland, Amsterdam, 1971b.

Carleton, W.J., "Linear Programming and Capital Budgeting Models: A New Interpretation," Journal of Finance, Vol. XXIV, No. 5 (December, 1969), pp. 825-833.

Charnes, A., W.W. Cooper, and M.H. Miller, "Application of Linear Programming to Financial Budgeting and the Costing of Funds," Journal of Business, Vol. XXXII, No. 1 (January, 1959), pp. 20-46.

Davis, R.E., D.A. Kendrick, and M. Weitzman, "A Branch-and-Bound Algorithm for Zero-One Mixed Integer Programming Problems," Development Economic Report No. 69, Center for International Affairs, Harvard University, 1967.

Dean, J., Capital Budgeting, Columbia University Press, New York, 1951.

Driebeek, N.J., "An Algorithm for the Solution of Mixed Integer Programming Problems," Management Science, Vol. 12, No. 7 (March, 1967), pp. 576-587.

Driebeek, N.J., Applied Linear Programming, Addison-Wesley, Reading, Mass., 1969.

Everett, H., III, "Generalized Lagrange Multiplier Method for Solving Problems of Optimal Allocation of Resources," Operations Research, Vol. 11, No. 3 (May-June, 1963), pp. 399-417.

Fisher, I., The Theory of Interest, Macmillan, New York, 1930.

Garfinkel, R.S. and G.L. Nemhauser, Integer Programming, Wiley, New York, 1972.

Geoffrion, A.M., "Integer Programming by Implicit Enumeration and Balas' Method," SIAM Review, Vol. 9, No. 2 (April, 1967), pp. 178-190.

Geoffrion, A.M., "An Improved Implicit Enumeration Approach for Integer Programming," Operations Research, Vol. 17, No. 3 (May-June, 1969), pp. 437-454.

Geoffrion, A.M., and R.E. Marsten, "Integer Programming Algorithms: A Framework and State-of-the-Art Survey," Management Science, Vol. 18, No. 9 (May, 1972), pp. 465-491.

Geoffrion, A.M., "Lagrangean Relaxation and It's Uses in Integer Programming," Working Paper No. 195, Western Management Science Institute, UCLA, December, 1972.

Glover, F., "A Multi-Phase Dual Algorithm for the Zero-One Integer Programming Problem," Operations Research, Vol. 13, No. 6 (November-December, 1965), pp. 879-919.

Haldi, J., "Twenty-Five Integer Programming Test Problems," Working Paper No. 43, Graduate School of Busines, Stanford University, December, 1964.

Hillier, F.S., "Derivation of Probabilistic Information for the Evaluation of Risky Investments," Management Science, Vol. 9, No. 3 (April, 1963), pp. 443-457.

Hillier, F.S., "Supplement to the Derivation of Probabilistic Information for the Evaluation of Risky Investments," Management Science, Vol. 11, No. 3 (January, 1965), pp. 485-487.

Hillier, F.S., "Chance-Constrained Programming with 0-1 or Bounded Continuous Decision Variables," Management Science, Vol. 14, No. 1 (September, 1967), pp. 34-51.

Hillier, F.S., and G.J. Lieberman, Introduction to Operations Research, Holden-Day, San Francisco, 1967.

Hillier, F.S., The Evaluation of Risky Interrelated Investments, North-Holland, Amsterdam, 1969.

Hirshleifer, J., "On the Theory of Optimal Investment Decision," Journal of Political Economy, Vol. LXVI, No. 4 (August, 1958), pp. 329-352.

Klevorick, A.K., "Capital Budgeting Under Risk: A Mathematical Programming Approach," Econometric Research Program, Research Memorandum No. 89, Princeton University, September, 1969.

Lorie, J., and L.J. Savage, "Three Problems in Rationing Capital," Journal of Business, Vol. XXVIII, No. 4 (October, 1955), pp. 229-239.

Lusztig, P., and B. Schwab, "A Note on the Application of Linear Programming to Capital Budgeting," Journal of Finance and Quantitative Analysis, Vol. 3, No. 4 (December, 1968), pp. 426-431.

Mao, J.C.T., Quantitative Analysis of Financial Decisions, Macmillan, New York, 1969.

Markowitz, H.M., Portfolio Selection, Wiley, New York, 1959.

Naslund, B., Decisions Under Risk, Economic Research Institute, Stockholm, 1967.

Naslund, B., "A Model of Capital Budgeting Under Risk," Chapter 2 in R.F. Byrne, et al.,(ed.), Studies in Budgeting, North-Holland, Amsterdam, 1971.

Petersen, C.C., "Computational Experience with Variants of the Balas Algorithm Applied to the Selection of R&D Projects," Management Science, Vol. 13, No. 9 (May, 1967), pp. 736-750.

Roy, B., R. Benayoun, and J. Tergny, "From S.E.P. Procedure to the Mixed Ophelie Program," in J. Abadie (ed.), Integer and Nonlinear Programming, North-Holland, Amsterdam, 1970.

Schrage, L., "Using Decomposition in Integer Programming," Naval Research Logistics Quarterly, Vol. 20, No. 3 (September, 1973), pp. 469-476.

Seppälä, Y., "Constructing Sets of Uniformly Tighter Linear Approximations for a Chance Constraint," Management Science, Vol. 17, No. 11 (July, 1971), pp. 736-749.

Tomlin, J.A., "Branch-and-Bound Methods for Integer and Non-Convex Programming," in J. Abadie (ed.), Integer and Nonlinear Programming, North-Holland, Amsterdam, 1970.

Tomlin, J.A., "An Improved Branch-and-Bound Method for Integer Programming," Operations Research, Vol. 19, No. 4 (July-August, 1971), pp. 1070-1075.

Tuan, N. Ph., "A Flexible Tree-Search Method for Integer Programming Problems," Operations Research, Vol. 19, No. 1 (January-February, 1971), pp. 115-119.

Weingartner, H.M., "Capital Budgeting of Interrelated Projects: Survey and Synthesis," Management Science, Vol. 12, No. 7 (March, 1966a), pp. 485-516. Also in Weingartner [1967].

Weingartner, H.M., "Criteria for Programming Investment Project Selection," Journal of Industrial Economics, Vol. 15, No. 1 (November, 1966b), pp. 65-76. Also in Weingartner [1967].

Weingartner, H.M., Mathematical Programming and the Analysis of Capital Budgeting Problems, Markham, Chicago, 1967.

Weingartner, H.M., "Some New Views on the Payback Period and Capital Budgeting Decisions," Chapter 5 in R.F. Byrne, et al., (ed.), Studies in Budgeting, North-Holland, Amsterdam, 1971.

10 SEP 80 682

Thesis 163641
R2545 Reardon
Decomposition of dual-angular integer programs, with applications to multi-stage capital budgeting.
10 SEP 80 DISPLAY 682

163641
Thesis
R2545 Reardon
Decomposition of dual-angular integer programs, with applications to multi-stage capital budgeting.

thesR2545
Decomposition of dual-angular integer pr
3 2768 001 01275 0
DUDLEY KNOX LIBRARY